Introduction to Differential Equations

Part I: Definitions and Classifications

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Definitions

A **Differential Equation** is an equation involving an unknown function of at least 1 variable and its derivatives.

The unknown function is commonly referred to as the dependent variable.

An **Ordinary Differential Equation (ODE)** is an equation containing functions of only 1 independent variable and its derivatives.

General Form

The general form of an ODE is given by:

Implicit Form

$$F(x, y, y', y'', \dots, y^{(n)}) = 0$$

Explicit Form

$$F(x, y, y', y'', \dots, y^{(n-1)}) = y^{(n)}$$

where:

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$y'' = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$$

$$y^{(n)} = \frac{\mathrm{d}^n y}{\mathrm{d}x^n}$$

Order

The **order** of an ODE is given by the highest order derivative of the unknown function (**dependent variable**).

Newton's 2^{nd} Law of Motion

$$F(t) = m * a(t)$$

$$F \equiv Force$$
$$m \equiv Mass$$
$$a \equiv Acceleration$$

However, remember that acceleration (a) is the first derivative of velocity (v), and velocity is the first derivative of position (x). Thus acceleration is the second derivative of position.

Mathematically,

$$a(t) = \frac{\mathrm{d}v(t)}{\mathrm{d}t} = v' = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathrm{d}x(t)}{\mathrm{d}t}) = \frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = x''$$
$$a(t) = x''$$

Newton's 2nd Law of Motion (continued)

After some rearranging, we can rewrite our original equation in several ways:

$\frac{F(t)}{m} = a(t)$	Explicit, 0th order ODE in a(t)
$\frac{F(t)}{m} = v(t)'$	Explicit, 1th order ODE in v(t)
$\frac{F(t)}{m} = x(t)''$	Explicit, 2th order ODE in x(t)

Autonomous ODEs

If the independent variable is equal to 0, the ODE is said to be autonomous.

Examples:

$F(y, y', \ldots, y^{(n)}) = 0$ Implicit, Autonomous ODE

$F(y, y', \ldots, y^{(n-1)}) = y^{(n)}$ Explicit, Autonomous ODE

Linear vs. Nonlinear

Linear: All derivatives are less than or equal to power 1

Nonlinear: At least 1 derivative has power greater than or equal to 2 or contains aa product.

Linear Example

$$y^{(n)} = \sum_{i=0}^{n-1} a_i(x) y^{(i)} + c(x)$$

$$a_i \equiv \textit{coefficients}$$

$$y^{(i)} \equiv \textit{derivatives}$$

$$c(x) \equiv \textit{constant}$$

F is a linear combination of all derivative terms.

If $c(x) = 0$ we call it a homogeneous linear ODE
If $c(x) \neq 0$ we call it a inhomogeneous linear ODE

Nonlinear Examples

$$y' = 2y^2 - y^3$$

$$y' = e^y$$

$$y' = \ln(y)$$

$$y' = \sin(y)$$

$$y'' + (y')^2 y = 0$$

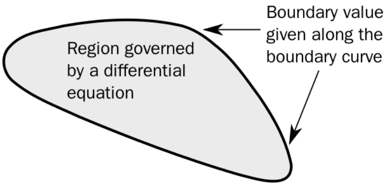
In each case, there is at least one y-term that is nonlinear.

IVP vs. BVP

Initial Value Problems: (1 Boundary)
An initial (starting) value for the dependent variable is given.
Thus the differential equation is an evolution.

$$y' = 3y;\ y(0) = 5$$

Boundary Value Problems: (2+ Boundaries)



Most real world problems include physical boundaries.

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