Introduction to Differential Equations

Part II: Case Study 1: Logarithmic Decay

Patrick Marsh

Mid-America Christian University

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The 3 Questions

In ODE there are 2 fundamental theorems, we must always be thinking about:

1. EXISTENCE
2. UNIQUENESS

Stemming from these two theorems (which we'll discuss in the coming days/weeks) when we come to a differential equation, we should get in the habit of asking ourselves the following three questions:

The 3 Questions (continued)

1. Does a solution exist? (Existence Theorem)
2. If a solution exists, is the solution unique or are there multiple solutions?* (Uniqueness Theorem)
3. Given at least one solution exists, can we find it?

*An addedum to question 2 might be:
Under what conditions is a solution guaranteed to be unique?

Logarithmic Decay

Radioactive isotopes like C_{14} decay with a time according to

$$y(t) = e^{-\lambda t}$$

Initially at time $t = 0$ or t_0 then $y(t) = y_0$. This is the initial number of atoms. After a time, $t_{\frac{1}{2}} = \frac{\ln(2)}{\lambda}$, then $y(t) = \frac{y_0}{2}$. This time is called the half-life, because the number of atoms has decreased by $\frac{1}{2}$. The λ is called the decay constant and is different for each radioactive isotope. For C_{14}, it's half life is approximately 5,730 years.

Logarithmic Decay (continued. . . 2)

If we differentiate, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}[y(t) = y_0 e^{-\lambda t}]$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}[y_0 e^{-\lambda t}]$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = y_0 \frac{\mathrm{d}}{\mathrm{d}t}[e^{-\lambda t}]$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -\lambda y_0 e^{-\lambda t}$$

But recall that $y(t) = e^{-\lambda t}$ so

$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -\lambda y$ Linear, 1^{st} order, Explicit ODE

Logarithmic Decay (continued. . . 3)

So we have:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -\lambda y$$

In words, this means the rate of change ($\frac{\mathrm{d}y}{\mathrm{d}t}$) of the number of atoms is inversely proportional to the nuber of atoms initially present (y).

To solve this ODE,we first note that is is separable. We will divide each side by y and multiply each side by $\mathrm{d}t$. After doing this, we have

$\frac{\mathrm{d}y(t)}{y} = -\lambda \, \mathrm{d}t$ This is called the Differential Form

Logarithmic Decay (continued. . . 4)

Now let's integrate both sides:

$$\int \frac{\mathrm{d}y}{y} = \int -\lambda \, \mathrm{d}t$$

$$\ln(y) = -\lambda \int \mathrm{d}t$$

$$\ln(y) = -\lambda t + C$$

$$y = e^{-\lambda t + C}$$

$$y = e^{-\lambda t} \cdot e^{C}$$

Logarithmic Decay (continued. . . 5)

$$y = e^{-\lambda t} \cdot y_0$$

$$y = y_0 e^{-\lambda t}$$

$$\xrightarrow{\text{Differentiate}}$$

$$\underset{\text{Solution}}{y(t) = e^{-\lambda t}} \qquad \underset{\text{ODE}}{\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -\lambda y}$$

$$\xleftarrow[\text{Integrate}]{}$$

We see that for this solution, the ODE and its solution are the calculus inverses of each other!

Logarithmic Decay (continued. . . 6)

The solution means that if we have 1024 atoms of C_{14} to begin with as time progresses we have:

$$1024 \longrightarrow 512 \longrightarrow 128 \longrightarrow \ldots \longrightarrow 4 \longrightarrow 2 \longrightarrow 1$$

It will take $\log_2(1024)$ steps to decay. This is $5730 \cdot log_2(1024) = 5730 \cdot 10 = 57,300$ years to decay!

Integration

We saw that to solve the differential equation for logarithmic decay, we first put the problem in differential form, integrated, and did a little algebra to get the solution.

Integration

You need to know the following standard integrals!
(There will be a quiz!)

1. $\int \frac{\mathrm{d}x}{x} = \ln|x| + C$
2. $\int \frac{a}{bx+k}\,\mathrm{d}x = \frac{a}{b}\ln|bx+k| + C$
3. $\int \sin(x)\,\mathrm{d}x = -\cos(x) + C$
4. $\int \cos(x)\,\mathrm{d}x = \sin(x) + C$
5. $\int \sin(ax)\,\mathrm{d}x = -\frac{1}{a}\cos(ax) + C$
6. $\int \cos(ax)\,\mathrm{d}x = \frac{1}{a}\sin(ax) + C$
7. $\int \mathrm{d}x = x + C$
8. $\int x^n\,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + C$ $(n \neq -1)$
9. $\int e^x\,\mathrm{d}x = e^x + C$
10. $\int e^{ax}\,\mathrm{d}x = \frac{1}{a}e^{ax} + C$
11. $\int \ln(x)\,\mathrm{d}x = x\ln|x| - x + C$

Integration

It will help if you remember/know the following:

- U-Substitution
  - For use in the arguments of functions (e.g., $\sin(g(x)), e^{g(x)}$)
- Method of Partial Fractions
  - For integrals of fraction functions
- Tabular Integration by Parts
  - For integrals of the product of 2 functions
- Rationalize denominators that involve roots

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