Separable Differential Equations

Part II: How to Solve

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Solving Separable ODEs

Consider the following ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t}{y^2}$$

What happens if we try to solve this equation by simply integrating both sides of the equation?

Solving Separable ODEs

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t}{y^2}$$

$$\int \frac{\mathrm{d}y}{\mathrm{d}t} = \int \frac{t}{y^2}$$

$$y(t) = \int \frac{t}{y^2}$$

...and now we're stuck...
...since we don't know what the function y^2 is...

Informal Algebra (aka "The Trick")

What if we multiplied each side by $(y^2 \, \mathrm{d}t)$ before integrating?

Informal Algebra (aka "The Trick")

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t}{y^2}\right) y^2 \mathrm{d}t$$

$$y^2 \, \mathrm{d}y = t \, \mathrm{d}t$$

$$\int y^2 \, \mathrm{d}y = \int t \, \mathrm{d}t$$

$$\frac{y^3}{3} = \frac{t^2}{2} + C$$

$$y(t) = \left(\frac{3t^2}{2} + 3C\right)^{1/3} \iff y(t) = \left(\frac{3t^2}{2} + k\right)^{1/3}$$

Informal Algebra: What's Really Going On

Begin with a separable differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(t)h(y)$$

We then divided both sides by $h(y)$:

$$\frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}t} = g(t)$$

However, y is actually a function of t so we really should write it as $y(t)$:

$$\frac{1}{h(y(t))}\frac{\mathrm{d}y}{\mathrm{d}t} = g(t)$$

Informal Algebra: What's Really Going On

Since both sides have a function of t in them, we can integrate with respect to t:

$$\int \frac{1}{h(y(t))} \frac{\mathrm{d}y}{\mathrm{d}t} \mathrm{d}t = \int g(t) \mathrm{d}t$$

Here's the catch. We can make a "u-substitution" by replacing the function $y(t)$ by a new variable y. Since we changed the function via "u-substitution" we must also change the derivative appropriately. In our original equation, the derivative of $y(t)$ is $\frac{\mathrm{d}y}{\mathrm{d}t}$ so in our new equation we must replace $\frac{\mathrm{d}y}{\mathrm{d}t}$ with the derivate of y, which is merely $\mathrm{d}y$. Therefore the left-hand side is transformed the following way:

$$\int \frac{1}{h(y(t))} \frac{\mathrm{d}y}{\mathrm{d}t} \mathrm{d}t = \int \frac{1}{h(y)} \mathrm{d}y$$

Informal Algebra: What's Really Going On

Therefore, we can use the previous equation to rewrite our separable differential equation as follows:

$$\int \frac{1}{h(y(t))} \mathrm{d}y = \int g(t) \mathrm{d}t$$

Which is exactly what we wanted!

The General Solution

The general solution to a differential equation is a solution we can use to solve any initial-value problem. Typically this is achieved by leaving the constant of integration in the solution versus solving for a specific value for the constant of integration.

In other words, the general solution consists of ALL solutions!

The Case of the Missing Solution(s)

Missing Solutions

We must be careful when determining the general solution!
For example:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2$$

This ODE is autonomous and therefore separable.

$$\int \frac{\mathrm{d}y}{y^2} = \int \mathrm{d}t$$

$$-\frac{1}{y} = t + C$$

$$y(t) = -\frac{1}{t + C}$$

Missing Solutions

We have found the general solution of

$$y(t) = -\frac{1}{t + C}$$

...or have we...

Missing Solutions

What happens if we have the initial condition

$$y(0) = 0$$

Our solution becomes

$$y(0) = -\frac{1}{C}$$

$$0 = -\frac{1}{C}$$

which is unsolvable!

Missing Solutions

This means the solution to our ODE using the separation of variables technique does **NOT**give us every solution. It fails to give us the trivial solution:

$$y(t) = 0$$

which is satisfied for all t

Missing Solutions

Thus, when asked to give the general solution, be mindful for potential pitfalls such as this. In this case the general solution would include *both*

$$y(t) = -\frac{1}{t+C} \text{ and } y(t) = 0$$

Getting Stuck

Just because a differential equation is separable, doesn't guarantee that we can solve it without the aid of computers. For example:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y}{1+y^2}$$

$$\left(\frac{1+y^2}{y}\right)\mathrm{d}y = \mathrm{d}t$$

$$\int \left(\frac{1+y^2}{y}\right)\mathrm{d}y = \int \mathrm{d}t$$

Getting Stuck

$$\int (\frac{1}{y} + y)\,\mathrm{d}y = \int \mathrm{d}t$$

$$\int (\frac{1}{y})\,\mathrm{d}y + \int y\,\mathrm{d}y = \int \mathrm{d}t$$

$$\ln|y| + \frac{y^2}{2} = t + C$$

...and now we're stuck...

Getting Stuck

We cannot derive an explicit formula for y.
However, we do have an implicit form.

Getting Stuck - Another Way

Suppose we have the ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sec(y^2)$$

Separating and integrating we get...

$$\int \frac{1}{\sec(y^2)}\,\mathrm{d}y = \int \mathrm{d}t$$

Which is the same as:

$$\int \cos(y^2)\,\mathrm{d}y = \int \mathrm{d}t$$

Which extreme difficult to solve!

Getting Stuck - Another Way

The lesson is that even if an ODE is separable, carrying out the required algebra and integration can be extremely difficult, or even impossible!