Qualitative Techniques:

Slope Fields

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The Geometry of $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$

If the function $y(t)$ is a solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$ and if its graph passes through the point (t_1, y_1) where $y_1 = y(t)$, then the differential equation says that the derivative $\frac{\mathrm{d}y}{\mathrm{d}t}$ at $t = t_1$ is given by the number $f(t_1, y_1)$.

Geometrically, this equality of $\frac{\mathrm{d}y}{\mathrm{d}t}$ at $t = t_1$ with $f(t_1, y_1)$ means that the slope of the tangent line to the graph of $y(t)$ at the point (t_1, y_1) is $f(t_1, y_1)$.

The Geometry of $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$

There is nothing special about (t_1, y_1) other than it is a point on the line of the solution.

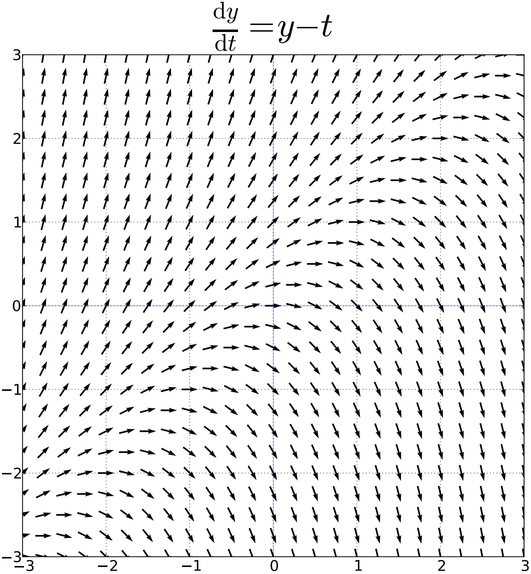
Generally speaking, the values of $f(t, y)$ yield the slopes of the tangents at all points on the graph of $y(t)$.

Thus, if we are given a function $f(t, y)$, we can obtain a rough idea of the graphs of the solutions by sketching its corresponding slope field.

The Algorithm

1. Choose a point (t, y)
2. Make sure your ODE is of the form $\frac{dy}{dt} = f(t, y)$
3. Substitute your point (t, y) into the ODE. In other words, replace t in the ODE with the t-value of your point and do the same for the y-value.
4. The resulting value is the slope at point (t, y)
5. Repeat this for more points until a general idea of the solution begins to emerge.

Example Slope Field



Important Special Cases

As we've discussed ODEs of the form

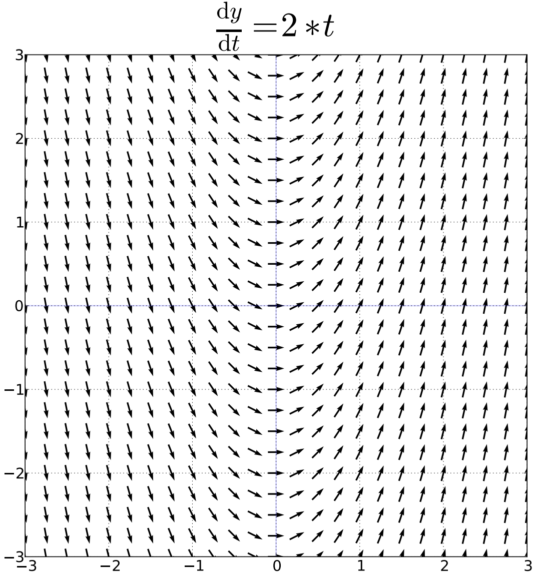
$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t) \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$$

are somewhat easier to solve analytically because they are separable. The geometry of their slope fields are also special.

Slope Fields for $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$

In ODEs of this form, the slope at any point is the same as the slope of any other point with the same t-coordinate. Geometrically, this implies that all of the slope marks are parallel on each vertical line throughout the domain. Conversely, when the slope marks of a slope field are all parallel in the vertical, you know the ODE is of the form $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$.

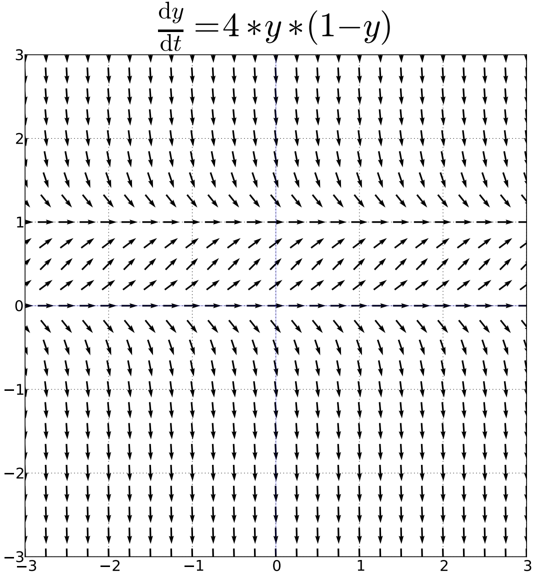
Slope Fields for $\frac{dy}{dt} = f(t)$



Slope Fields for $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$

Similar to ODEs of the form $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$, in which the slope marks are parallel in the vertical, ODEs of the form $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$ have slope marks that are parallel in the horizontal.

Slope Fields for $\frac{dy}{dt} = f(t)$



Why Do We Care

So why do we care about visual solutions? Well, remember, sometimes we can't easily find an analytic solution. For example:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = e^{\frac{y^2}{10}} \sin^2 y$$

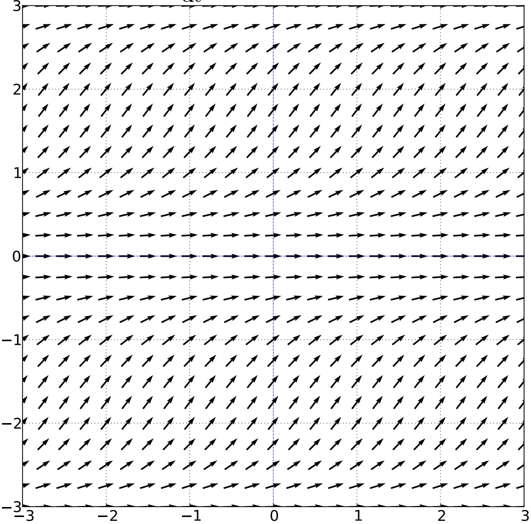
Separating we get

$$\int \frac{\mathrm{d}y}{e^{\frac{y^2}{10}} \sin^2 y} = \int \mathrm{d}t$$

which is very hard to solve. However, we can get a sense of the behavior of the solution by graphic the slope field.

Why Do We Care

$$\frac{\mathrm{d}y}{\mathrm{d}t} = e^{\frac{y^2}{10}} \sin^2 y$$



Take Away Point

When only knowledge of the qualitative behavior of the solution is required, sketches of solutions obtained from slope fields can sometimes suffice. In other applications, it is necessary to know the exact value (or almost exact value) of the solution with a given initial condition. In these situations analytic and/or numerical methods can't be avoided. But even then, it is nice to have a picture of what solutions look like.