Linear Differential Equations

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Review - Linear ODEs

A first order Ordinary Differential Equation is linear if it can be written in the form:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a(t)y + b(t)$$

where $a(t)$ and $b(t)$ are arbitrary functions of t.

Review - Linear ODEs

Sometimes you must do some algebra in order to "see" that an ODE is linear. The following doesn't appear to be a linear ODE

$$\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = ty + 2$$

but it can be rewritten as:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ty + 3y + 2$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(t + 3) + 2$$

which is easily seen to be a linear ODE.

Review - Linear ODEs

The term linear refers to the fact that the dependent variable appears in the equation only to the first power.

Review - Homogeneous vs. Nonhomogeneous

Recall our standard linear ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a(t)y + b(t)$$

If $b(t) = 0$ for all t, then the equation is said to be homogeneous. Otherwise, the equation is said to be

nonhomogeneous.

Linearity Principle

There are two linearity principles, one for homogeneous equations and different one for nonhomogeneous equations.

Linearity Principle - The Homogeneous Case

If $y_h(t)$ is a solution to the homogeneous linear equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a(t)y$$

then an constant multiple of $y_h(t)$ is also a solution. That is, $\kappa y_h(t)$ is a solution for any constant κ.

This is known as the "Linearity Principle".

Linearity Principle - The Nonhomogeneous Case

The Linearity Principle does not hold for nonhomogeneous liner equations. However, there is a nice relationship between nonhomogeneous linear equations and their associated homogeneous equation.

Linearity Principle - The Nonhomogeneous Case

Consider the nonhomogeneous equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a(t)y + b(t)$$

and its associated homogeneous equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a(t)y$$

1. If $y_h(t)$ is any solution of the homogeneous equation and $y_p(t)$ is any solution of the nonhomogeneous equation ("p" stands for particular) then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.
2. Suppose $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation, then $y_p(t) - y_q(t)$ is a solution of the associated homogeneous equation.

Solving Linear Equations

We have a 3-step technique t find the general solution to linear equations:

1. Find the solution to the associated homogeneous equation, $y_h(t)$.
2. Find one of the particular solutions to the nonhomogeneous equation, $y_p(t)$.
3. Sum the two together, $y_h(t) + y_p(t)$

"Lucky Guess Technique"

Consider the linear differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2y + e^t$$

The associated homogeneous differential equation is $\frac{\mathrm{d}y}{\mathrm{d}t} = -2y$ with a solution of $y_h(t) = \kappa e^{-2t}$

Now we need to solve the nonhomogeneous differential equation. As the name suggests, the "Lucky Guess Technique" suggests we try and guess a solution. To help us with our guess, rewrite the equation so all "y" terms are on a side by themselves.

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = e^t$$

"Lucky Guess Technique"

Now, we need to guess something for y such that when we plug it in on the left and simplify, e^t is the result. We probably should not guess sin or cos since that will leave us with trig functions. We also probably should not guess polynomials because that will not get us e^t. Thus, let's try e^t since its derivative is e^t.

“Lucky Guess Technique”

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2y + e^t$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^t) + 2(e^t) = e^t$$

$$e^t + 2e^t = e^t$$

$$3e^t \neq e^t$$

Close, but not quite right...

“Lucky Guess Technique”

We were only off be a factor of 3, so perhaps we should guess a constant multiple of e^t? Let’s have the ODE tell us what that constant should be …

Method of Undetermined Coefficients

We tried $y_p(t) = e^t$, but were slightly off. Now, let's try $y_p(t) = \alpha e^t$, where α is a constant to be defined later.

This technique is known as the Method of Undetermined Coefficients.

Method of Undetermined Coefficients

Let's plug in our new guess:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha e^t) + 2(\alpha e^t) = e^t$$

$$\alpha e^t + 2\alpha e^t = e^t$$

$$3\alpha e^t = e^t$$

$$3\alpha = 1$$

$$\alpha = \frac{1}{3}$$

Method of Undetermined Coefficients

Thus, our particular solution is:

$$y_p(t) = \frac{1}{3}e^t$$

This means that our general solution becomes

$$y(t) = y_h(t) + y_p(t)$$

$$y(t) = \kappa e^{-2t} + \frac{1}{3}e^t$$

Another Lucky Guess

Now let's try a nonhomogeneous equation with trig functions. . .

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = \cos 3t$$

This is the same as the last example, except for $b(t)$. Thus the solution to the associated homogeneous part is the same as last time. Namely, $y_h(t) = \kappa e^{-2t}$

Since we have a trig function for $b(t)$, our "guess" should be something other than e^t! Let's try:

$$y_p(t) = \alpha \cos 3t + \beta \sin 3t$$

Why Both cos & sin

Because simpler guesses with only a sin or cos will always fail since we'll end up with sin and cos when we plug into our left-hand side. Using both, we can do some simplification. . .

Another Lucky Guess

To determine α and β, substitute back into our equation...

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = \cos 3t$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha \cos 3t + \beta \sin 3t) + 2(\alpha \cos 3t + \beta \sin 3t) = \cos 3t$$

$$3\alpha \sin 3t + 3\beta \cos 3t + 2\alpha \cos 3t + 2\beta \sin 3t = \cos 3t$$

$$(-3\alpha + 2\beta) \sin 3t + (3\beta + 2\alpha) \cos 3t = \cos 3t$$

Another Lucky Guess

Since we need to get rid of the sin and keep the cos on the left-hand side (in order to match the right-hand side), we need the coefficient in front of the sin term to be 0 while keeping the coefficient of the cos term equal to 1.

To do this, we solve the following system of equations:

$$-3\alpha + 2\beta = 0 \quad (1)$$
$$2\alpha + 3\beta = 1 \quad (2)$$

To solve this system of equations, multiply (1) by 2 and (2) by 3.

Another Lucky Guess

After doing this we are left with:

$$-6\alpha + 4\beta = 0$$
$$6\alpha + 9\beta = 3$$

Combining the two equations, we are left with

$$13\beta = 3$$

$$\beta = \frac{3}{13}$$

Another Lucky Guess

To solve for α, plug β back into either (1) or (2) and simplify.

$$2\alpha + 3\beta = 1$$

$$2\alpha + 3(\frac{3}{13}) = 1$$

$$2\alpha + \frac{9}{13} = 1$$

$$2\alpha = \frac{4}{13}$$

$$\alpha = \frac{2}{13}$$

Another Lucky Guess

Plugging α and β back into our guess, we find a particular solution.

$$y_p(t) = \frac{2}{13}\cos 3t + \frac{3}{13}\sin 3t$$

And the general solution becomes

$$y(t) = y_h(t) + y_p(t)$$

$$y(t) = \kappa e^{-2t} + \frac{2}{13}\cos 3t + \frac{3}{13}\sin 3t$$

Second Guesses

Sometimes our first guess may not work, no matter how reasonable it may be. If this happens, simply guess again. Consider the following:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2y + 3e^{-2t}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 3e^{-2t}$$

This is simply the last differential with yet another $b(t)$ term. So, we, once again, have the same solution to the associated homogeneous equation, $y_h(t) = \kappa e^{-2t}$.

Second Guesses

Our first guess, using the Method of Undetermined Coefficients, should be $y_p(t) = \alpha e^{-2t}$. So, we plug it in:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha e^{-2t}) + 2(\alpha e^{-2t}) = 3e^{-2t}$$

$$-2\alpha e^{-2t} + 2\alpha e^{-2t} = 3e^{-2t}$$

$$0 = 3e^{-2t}$$

We have a problem...

Second Guesses

No matter how we pick our coefficient α we will always get zero. This is because our guess is turns out to be one solution to the associated homogeneous part of the ODE.

Our guess must contain a factor of e^{-2t} to have any hope of being a solution. Unfortunately, there are a wide variety of possible choices. We need a second guess for $y_p(t)$ that contains an e^{-2t} term, is not a solution to the homogeneous equation, and is as simple as possible.

Second Guesses - Product Rule to the Rescue

Since we can't multiply e^{-2t} by a constant, how about another variable?. We can multiply e^{-2t} by t and exploit the Product Rule! So, our guess becomes $y_p(t) = \alpha t e^{-2t}$.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha t e^{-2t}) + 2(\alpha t e^{-2t}) = 3e^{-2t}$$

$$-2\alpha t e^{-2t} + \alpha e^{-2t} + 2\alpha t e^{-2t} = 3e^{-2t}$$

$$\alpha e^{-2t} = 3e^{-2t}$$

$$\alpha = 3$$

Second Guesses - Product Rule to the Rescue

Thus, one particular solution is $y_p(t) = 3te^{-2t}$.

And the general solution becomes:

$$y(t) = y_h(t) + y_p(t)$$

$$y(t) = \kappa e^{-2t} + 3te^{-2t}$$