

# Integrating Factors

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# Integrating Factors

Given a nonhomogeneous linear differential equation

$$\frac{dy}{dt} = a(t)y + b(t)$$

rewrite it as

$$\frac{dy}{dt} + g(t)y = b(t)$$

where  $g(t) = -a(t)$

# Integrating Factors

If we stare at the left-hand side long enough (mathematicians did this for decades), we might begin to notice that it looks like something...

...in fact we mentioned this at the end of the last lecture ...

It looks like the Product Rule!

# Integrating Factors

Specifically, the Product Rule states that the derivative of the product of  $y(t)$  and a function  $\mu(t)$  is ...

$$\frac{d(\mu(t)y(t))}{dt} = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t)$$

Note that one term on the right-hand side has  $\frac{dy}{dt}$  in it and the other term has  $y$  in it, just like the left-hand side of our nonhomogeneous linear equation.

# Integrating Factors

Here's the clever part . . . Multiply both sides of the original differential equation by an (as of now) unspecified function  $\mu(t)$ . We then obtain a new differential equation.

$$\frac{dy}{dt} + g(t)y = b(t)$$
$$\mu(t)\frac{dy}{dt} + \mu(t)g(t)y = \mu(t)b(t)$$

Now the left side looks even more like the Product Rule!

# Integrating Factors

For the moment, let's assume that we do have a function  $\mu(t)$  so that the left-hand side actually is the derivative of the product  $\mu(t)y(t)$ . That is, suppose we have a function  $\mu(t)$  that satisfies

$$\frac{d(\mu(t)y(t))}{dt} = \mu(t)\frac{dy}{dt} + \mu(t)g(t)y$$

This would mean that we could rewrite our original differential equation as

$$\frac{d(\mu(t)y(t))}{dt} = \mu(t)b(t)$$

# Integrating Factors

Why is this important? Because we can now integrate both sides of the equation with respect to  $t$  to obtain

$$\mu(t)y(t) = \int \mu(t)b(t)dt$$

and consequently

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)b(t)dt$$

Thus, if we can find a function  $\mu(t)$  and can solve the integral  $\int \mu(t)b(t)dt$  we know our solution.

# Finding The Integrating Factors

The previous derivation of  $y(t)$  hinges on one big assumption . . .

. . . that we can find  $\mu(t)$  . . .

## Finding The Integrating Factors

How do we determine  $\mu(t)$ ?

Apply the Product Rule to the left-hand side.

$$\frac{d(\mu(t)y(t))}{dt} = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t)$$

Now, we need to set this result equal to the left-hand side of the original differential equation that was multiplied by  $\mu$ .

$$\mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t) = \mu(t)\frac{dy}{dt} + \mu(t)g(t)y(t)$$

Canceling the  $\mu(t)\frac{dy}{dt}$  terms on both sides, we're left with

$$\frac{d\mu}{dt}y(t) = \mu(t)g(t)y(t)$$

# Finding The Integrating Factors

If we cancel  $y(t)$  we can see that if we find a function that satisfies

$$\frac{d\mu}{dt} = \mu(t)g(t)$$

we'll find the correct  $\mu(t)$

# Finding The Integrating Factors

The good news is that this last equation is just

$$\frac{d\mu}{dt} = \mu g(t)$$

which is a homogeneous linear differential equation!

Solving this ODE we find that

$$\mu(t) = e^{\int g(t)dt}$$

Note: when computing  $\mu(t)$  we get a constant of integration (+C) in the exponent. Since we only need one integrating factor to solve the equation, we choose the constant to be whatever is the most convenient. That is usually 0.

# Integrating Factors

The function  $\mu(t)$  is called an “**Integrating Factor**”. Thus, anytime we want to obtain an explicit solution to a nonhomogeneous, linear differential equation

$$\frac{dy}{dt} + g(t)y = b(t)$$

we must

- 1 Compute the integrating factor,  $\mu(t)$
- 2 Multiple the ODE by  $\mu(t)$
- 3 Integrate!

## Integrating Factors - An Example

$$\frac{dy}{dt} + \frac{2}{t}y = t - 1$$

First we need to compute the integrating factor ...

$$\mu(t) = e^{\int g(t)dt} = e^{\int \frac{2}{t}dt} = e^{2 \ln |t|} = e^{\ln t^2} = t^2$$

Remember, the idea behind this method is to multiply both sides of the original ODE by  $\mu(t)$  so the left-hand side of the new equation is the result of the Product Rule.

$$t^2 \cdot \left[ \frac{dy}{dt} + \frac{2}{t}y = t - 1 \right]$$

$$t^2 \frac{dy}{dt} + t^2 \frac{2}{t}y = t^3 - t^2$$

## Integrating Factors - An Example

Now, we reverse the Product Rule to make the left-hand side

$$\frac{d}{dt}(t^2y) = t^3 - t^2$$

Now, integrate with respect to  $t$

$$\int d(t^2y) = \int (t^3 - t^2)dt$$

$$t^2y = \frac{t^4}{4} - \frac{t^3}{3} + k$$

$$y = \frac{t^2}{4} - \frac{t}{3} + \frac{k}{t^2}$$

# Integrating Factors - An Example

Our general solution is

$$y = \frac{t^2}{4} - \frac{t}{3} + \frac{k}{t^2}$$

This problem is also good illustration of the Extended Linearity Principle.  $\frac{k}{t^2}$  is the general solution to the associated homogeneous equation, and  $\frac{t^2}{4} - \frac{t}{3}$  is one solution of the nonhomogeneous equation. Thus the general solution is the sum of these two solutions.

## Integrating Factors - Problems

Now try:

$$\frac{dy}{dt} - t^2y = t - 1$$

First, compute the integrating factor:

$$\mu(t) = e^{\int g(t)dt} = e^{\int -t^2y dt} = e^{-\frac{t^3}{3}}$$

Now, multiply both sides of the original ODE by  $\mu(t)$ .

$$e^{-\frac{t^3}{3}} \cdot \left[ \frac{dy}{dt} - t^2y = t - 1 \right]$$

$$e^{-\frac{t^3}{3}} \left( \frac{dy}{dt} \right) - e^{-\frac{t^3}{3}} (t^2y) = e^{-\frac{t^3}{3}} (t - 1)$$

## Integrating Factors - Problems

$$\frac{d}{dt}(e^{-\frac{t^3}{3}}y) = e^{-\frac{t^3}{3}}(t-1)$$

$$\int d(e^{-\frac{t^3}{3}}y) = \int e^{-\frac{t^3}{3}}(t-1)dt$$

$$e^{-\frac{t^3}{3}}y = \int e^{-\frac{t^3}{3}}(t-1)dt$$

$$y = \frac{1}{e^{-\frac{t^3}{3}}} \int e^{-\frac{t^3}{3}}(t-1)dt$$

... and we're stuck ...

## Integrating Factors - Problems

With this particular problem, we cannot find an explicit solution using integrating factors. This does not, however, mean that we can't use other techniques to find an explicit solution!