~~Sketch~~ The Geometry Of Systems

The Predator/Prey Vector Fields

Recall the Predator-prey system

$$\frac{dR}{dt} = 2R - 1.2RF$$

$$\frac{dF}{dt} = -F + 0.9RF$$

Let's consider the pair $(R(t), F(t))$ as a vector-valued function in the RF plane. For each t, we denote the vector $P(t)$

$$\vec{P}(t) = \begin{bmatrix} R(t) \\ F(t) \end{bmatrix}$$

The the vector-valued function $P(t)$ corresponds to the solutions curve $(R(t), F(t))$ in the RF plane. To compute the derivative of the vector valued function $P(t)$, we compute the derivatives of each component

$$\frac{d\vec{P}(t)}{dt} = \begin{bmatrix} \frac{dR}{dt} \\ \frac{dF}{dt} \end{bmatrix}$$

Using this notation, we can rewrite the predator-prey system as the single vector equation

$$\frac{d\vec{P}}{dt} = \begin{pmatrix} \frac{dR}{dt} \\ \frac{dF}{dt} \end{pmatrix} = \begin{bmatrix} 2R - 1.2RF \\ -F + 0.9RF \end{bmatrix}$$

At this point, we've done nothing more than introduce new notation.

The advantage of doing this will become clear as we consider the right hand side as a <u>vector field</u>.

The right hand side of the predator-prey system is a function that assigns a vector to each point in the RF-plane.

If we denote this function using the vector V, we have

$$\vec{V}\begin{pmatrix} R \\ F \end{pmatrix} = \begin{pmatrix} -2R - 1.2RF \\ -F + 0.9RF \end{pmatrix}$$

For example, at the point $(R,F) = (2,1)$

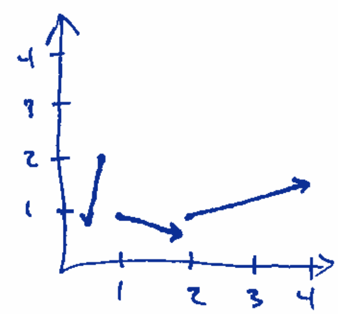
$$\vec{V}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2(2) - 1.2(2)(1) \\ -(1) + 0.9(2)(1) \end{pmatrix} = \begin{pmatrix} 1.6 \\ 0.8 \end{pmatrix}$$

$$\vec{V}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.6 \\ 0.8 \end{pmatrix}$$

$$\vec{V}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.8 \\ -0.1 \end{pmatrix}$$

$$\vec{V}\begin{pmatrix} 0.5 \\ 2.2 \end{pmatrix} = \begin{pmatrix} -0.32 \\ -1.21 \end{pmatrix}$$

$$\vdots$$



The use of vectors simplifies the notation. We can now write the predator-prey system as

$$\frac{d\vec{P}}{dt} = \vec{V}(\vec{P})$$

The vector rotation is more than a way to save ink. It also gives us a way to think about and to visualize systems of differential equations

The Vector-Field for a Simple Harmonic Oscillator

We can model the behavior of an undamped spring with the following

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

where $k \equiv$ spring constant
$m \equiv$ mass attached to spring
$y \equiv$ distance spring stretched from equilibrium
$t =$ time

Now, we don't want to deal with second order ODEs (at least not today) so we break it down into the following system

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -\frac{k}{m}y$$

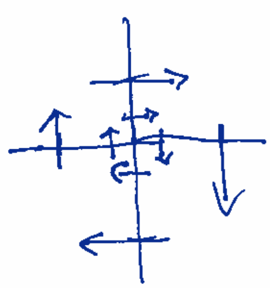
where $v = \frac{dy}{dt} \equiv$ velocity of the mass

In the special case where $\frac{k}{m} = 1$, we obtain the nice system

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -y$$

One reason this is so nice is that we obtain the vector field $\vec{F}(y,v) = (v,-y)$.
Every vector is merely the tangent line to the circle, rotating in the clockwise direction



Examples of Systems & Vector Fields

In general, for a system with two dependent variables of the form

$$\frac{dx}{dt} = f(x,y)$$

$$\frac{dy}{dt} = g(x,y)$$

we introduce the vector $\vec{Y}(t) = (x(t), y(t))$ and the vector field

$$\vec{F}(\vec{Y}) = \vec{F}(x,y) = (f(x,y), g(x,y))$$

With this notation, the system of two equations can be written in the compact form

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \vec{F}(\vec{Y})$$

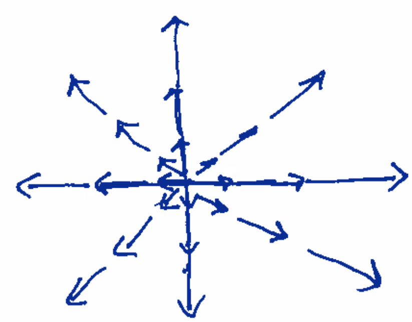
$$\frac{d\vec{Y}}{dt} = \vec{F}(\vec{Y})$$

Elementary Examples

$$\frac{dx}{dt} = x$$

$$\frac{dy}{dt} = y$$

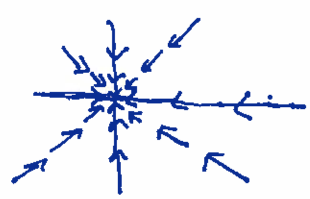
~~yields~~ yields the vector field $\vec{F}(x,y) = (x,y)$



$$\frac{dx}{dt} = -x$$

$$\frac{dy}{dt} = -y$$

yields the vector field $\vec{F}(x,y) = (-x,-y)$



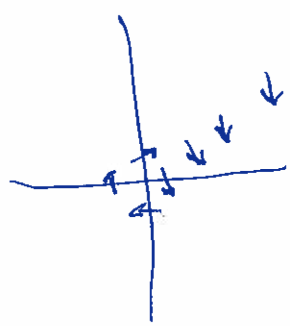
Geometry of Solutions

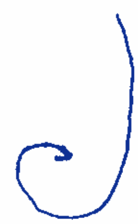
Once again consider

$$\frac{dx}{dt} = f(x,y) \qquad \Longrightarrow \qquad \frac{d\vec{Y}}{dt} = \vec{F}(\vec{Y})$$
$$\frac{dy}{dt} = g(x,y)$$

Interpreting this vector equation geometrically is the key to a geometric understanding of this ODE system. If we think of a solution $\vec{Y}(t) = (x(t), y(t))$ as a parameterized curve in the xy-plane, the $\frac{d\vec{Y}}{dt}$ yields the tangent vectors of the curve. Therefore the equation $\frac{d\vec{Y}}{dt} = \vec{F}(\vec{Y})$ says that the tangent vectors for the solution curves are given by the vectors in the vector field.

One consequence of this geometric interpretation is that we can go directly ~~for~~ from a sketch of a vector field $\vec{F}$ (or its direction field) to a sketch of the solution curves of the equation $\frac{d\vec{Y}}{dt} = \vec{F}(\vec{Y})$ without ever knowing the formula for $\vec{F}$.





Equilibrium Solutions

Consider
$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

DEFINITION

The point $\vec{Y}_0$ is an equilibrium point for the system $\frac{d\vec{Y}}{dt} = \vec{F}(\vec{Y})$ if $\vec{F}(\vec{Y}) = 0$. The constant function $\vec{Y}(t) = \vec{Y}_0$ is an equilibrium solution.

Equilibrium points are simply points where the right hand side of the system goes to 0. If $\vec{Y}_0$ is an equilibrium point, then the constant function

$$\vec{Y}(t) = \vec{Y}_0 \quad \text{for all } t$$

is a solution of the system. To verify this claim, note the constant function has $\frac{d\vec{Y}}{dt} = (0,0)$ for all t. On the other hand,

$\vec{F}(\vec{Y}(t)) = \vec{F}(\vec{Y}_0) = (0,0)$ at an equilibrium point. Hence, equilibrium points in the vector field correspond to constant solutions.

Computation of Equilibrium Points

Consider
$$\frac{dx}{dt} = 3x + y$$
$$\frac{dy}{dt} = x - y$$

has only one equilibrium point, the origin $(0,0)$. To see why, we simultaneously solve the two equations

(1) $3x + y = 0$
$x - y = 0$

(2) $3x + y = 0$
$3x - 3y = 0$

(3) $-2y = 0$

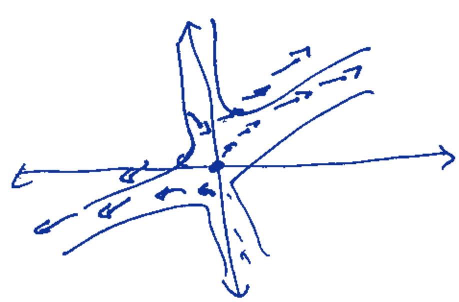
(4) $y = 0$

(5) $3x + 0 = 0$

(6) $3x = 0$

(7) $x = 0$

Draw the vector field and we see that the vectors are short near the origin



In all the systems considered so far, the independent variable has not appeared on the right hand side. Systems with this property are said to be autonomos. The word autonomous means self-governing, and roughly speaking, an autonomous system is self governing because it evolves according to differential equation that are determined entirely by the values of the dependent variables. An important geometric consequence is that the vector field associated with an autonomous system only depends on the dependent variables. As a result we need not worry about the independent variable when we sketch the vector field, the solution curves, and the phase portrait.

Although we will continue to focus on autonomous systems for the next few classes, many important systems are nonautonomous.