Analytic Methods For Special Systems

~~[illegible]~~

Previously, we've seen that we can sometimes derive a formula for the general solution of a differential equation, if that differential equation had a special form.

For systems of differential equations, the special forms for which we can apply analytic techniques to find explicit solutions are few and far between.

Today we'll examine some analytic techniques that apply to linear systems.

FIRST: CHECKING SOLUTIONS

As noted, finding formulas for a solution of a system can range from difficult to impossible. However, assume for a moment that someone gave us a potential solution. We can (relatively) easily check the solution. This is important because often times the techniques for solving linear systems is just a sophisticated guess.

Consider
$$\frac{dx}{dt} = -x + y$$
$$\frac{dy}{dt} = -3x - 5y$$

we can rewrite as a vector
$$\frac{d\vec{Y}}{dt} = \vec{F}(\vec{Y})$$

where $\vec{Y} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

and $\vec{F}(x,y) = \begin{bmatrix} -x+y \\ -3x-5y \end{bmatrix}$

now, suppose someone says

$$\vec{Y}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{-4t} - 3e^{-2t} \\ -3e^{-4t} + 3e^{-2t} \end{bmatrix}$$ is a solution

To verify that this is a solution, we must first compute the derivatives.

$$\frac{dx}{dt} = \frac{d}{dt}\left[e^{-4t} - 3e^{-2t}\right] = -4e^{-4t} + 6e^{-2t}$$

$$\frac{dy}{dt} = \frac{d}{dt}\left[-3e^{-4t} + 3e^{-2t}\right] = 12e^{-4t} - 6e^{-2t}$$

$$\therefore \quad \frac{dx}{dt} = -4e^{-4t} + 6e^{-2t}$$

$$\frac{dy}{dt} = 12e^{-4t} - 6e^{-2t}$$

Now, we must put our given solutions into the right hand side of our original equations

$$\left[\begin{aligned} \frac{dx}{dt} &= -x + y \\ \frac{dy}{dt} &= -3x - 5y \end{aligned}\right]$$

so

$$\begin{aligned} -x + y &= -1\left(e^{-4t} - 3e^{-2t}\right) + \left(-3e^{-4t} + 3e^{-2t}\right) \\ &= -e^{-4t} + 3e^{-2t} + (-3)e^{-4t} + 3e^{-2t} \\ &= \boxed{-4e^{-4t} + 6e^{-2t}} \end{aligned}$$

$$\begin{aligned} -3x - 5y &= -3\left(e^{-4t} - 3e^{-2t}\right) - 5\left(-3e^{-4t} + 3e^{-2t}\right) \\ &= -3e^{-4t} + 9e^{-2t} + 15e^{-4t} - 15e^{-2t} \\ &= \boxed{12e^{-4t} - 6e^{-2t}} \end{aligned}$$

$$-x + y = -4e^{-4t} + 6e^{-2t}$$

$$-3x - 5y = 12e^{-4t} - 6e^{-2t}$$

If $\frac{dx}{dt} = -x+y$ and $\frac{dy}{dt} = -3x-5y$

then the solution is correct.

recall, we have

$$\frac{dx}{dt} = -4e^{-4t} + 6e^{-2t} = -x+y$$

$$\therefore \quad \frac{dx}{dt} = -x+y \quad \checkmark$$

$$\frac{dy}{dt} = 12e^{-4t} - 6e^{-2t} = -3x-5y$$

$$\therefore \quad \frac{dy}{dt} = -3x-5y$$

and our provided solution is correct!

TRY: $\frac{dx}{dt} = 2x-y$

$\frac{dy}{dt} = x-2y$

now suppose someone says that the solution is

$$\vec{Y}(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}$$

To check...

$$\frac{d}{dt}\left(e^{-t}\right) = -e^{-t}$$

&

$$\frac{d}{dt}\left(3e^{-t}\right) = -3e^{-t}$$

now plug into right hand side

$$2x-y = 2\left(e^{-t}\right) - 3e^{-t}$$

$$= 2e^{-t} - 3e^{-t}$$

$$= -e^{-t}$$

$\Longrightarrow$

$$x-2y = e^{-t} - 2\left(3e^{-t}\right)$$

$$e^{-t} - 6e^{-t}$$

$$-5e^{-t}$$

Now, are things equal?

$$\frac{dx}{dt} = -e^{-t} = -e^{-t} = 2x - y$$

$$\frac{dy}{dt} = -3e^{-t} \neq -5e^{-t} = x - 2y$$

Thus, this is not the solution to the system!

Decoupled systems

One thing that makes systems of differential equations so difficult is that the rate of change of each ~~others~~ of the dependent variables often depends on the values of other dependent variables. However, sometimes there is not too much interdependence among the variables, and in that case we can often derive the general solution using previous techniques.

A system of differential equations is said to "decouple" if the rate of change of one or more of the dependent variables depends only on its own value.

Consider
$$\frac{dx}{dt} = -2x$$
$$\frac{dy}{dt} = -y$$

Since the equation for $\frac{dx}{dt}$ involves only x and the equation for $\frac{dy}{dt}$ involves only y, we can solve the two equations separately. When this happens we say the system is "completely decoupled". The general solution of

$\frac{dx}{dt} = -2x$ is $x(t) = K_1 e^{-2t}$ where $K_1 \equiv$ constant

$\frac{dy}{dt} = -y$ is $y(t) = K_2 e^{-t}$ where $K_2 \equiv$ any constant

we put these together to form the solution

$$\vec{Y}(t) = \begin{bmatrix} K_1 e^{-2t} \\ K_2 e^{-t} \end{bmatrix}$$

Partial Decoupled Systems

Consider
$$\frac{dx}{dt} = 2x + 3y$$
$$\frac{dy}{dt} = -4y$$

For this system, the rate of change of x depends on both x and y, but the rate of change of y depends only on y. We say that the dependent variable y "decouples" from the system and the system is "partially decoupled"

The general solution of the equation for y is $y(t) = k_2 e^{-4t}$. To solve for x, ~~so~~ substitute this expression for y into the equation for x.

$$\frac{dx}{dt} = 2x + 3y$$
$$\boxed{\frac{dx}{dt} = 2x + 3k_2 e^{-4t}}$$

This is a first order, linear equation which we can solve using methods we already know. From the Extended Linearity Principle, we know that we need just one particular solution of the nonhomogeneous equation, as well as the general solution of the associated homogeneous equation. To find $x_p(t)$ rewrite as

$$\frac{dx}{dt} - 2x = 3k_2 e^{-4t}$$

and guess a solution of the form $x_p(t) = \alpha e^{-4t}$

$$\frac{d}{dt}(\alpha e^{-4t}) - 2(\alpha e^{-4t}) = 3k_2 e^{-4t}$$
$$-4\alpha e^{-4t} - 2\alpha e^{-4t} = 3k_2 e^{-4t}$$
$$-6\alpha e^{-4t} = 3k_2 e^{-4t}$$
$$\boxed{\alpha = -\frac{1}{2}k_2}$$

(derivation Not shown) the solution to the associated homogeneous is $k_1 e^{2t}$ where k_1 is a const.

thus $x(t) = k_1 e^{2t} - \frac{1}{2}k_2 e^{-4t}$

We can put these two equations for $y(t)$ and $x(t)$ together

$$x(t) = K_1 e^{2t} - \frac{1}{2} K_2 e^{-4t}$$
$$y(t) = K_2 e^{-4t} .$$

The constants k_1 and k_2 can be adjusted to obtain any desired initial condition. For example, suppose we have

$$x(0) = 0$$
and $$y(0) = 1$$

To find the appropriate values of k_1 and k_2, substitute in $t=0$ and solve the resulting system.

$$x(0) = 0 = K_1 e^{(0)} - \frac{1}{2} K_2 e^{(0)}$$
$$y(0) = 1 = K_2 e^{(0)}$$

$$0 = K_1 - \frac{1}{2} K_2$$
$$1 = K_2$$

$$K_1 = \frac{1}{2} \quad \text{and} \quad K_2 = 1$$

$\therefore$

$$\boxed{\begin{aligned} x(t) &= \frac{1}{2} e^{2t} - \frac{1}{2} e^{-4t} \\ y(t) &= e^{-4t} \end{aligned}}$$