SIMPLE HARMONIC OSCILLATOR

THE MOTION OF A MASS ATTACHED TO A SPRING

Consider a mass that is attached to a spring and that slides along a frictionless plane. We want to understand its horizontal motion when the spring is stretched or compressed, and then released. In order to keep the model as simple as possible, we assume that the only force acting on the mass is the force of the spring. In other words, at this time, we are going to ignore air resistance and other forces that would dampen the motion of the mass.

There are two key quantities in this model – a quantity that measures the displacement of the mass from its natural rest position and the restoring force on the mass caused by the spring. We wish to determine the position of the mass as a function of time, so let $y(t)$ denote the position of the mass at time t. It is convenient to let $y=0$ represent the rest position of the mass. At rest, the spring is neither stretched, nor compressed, and it exherts no force on the mass. We'll use the convention that $y(t) < 0$ if the spring is compressed and $y(t) > 0$ if the spring is stretched.

Remember for Physics (or High School Science)

$$\text{Force} = \text{mass} \times \text{acceleration}$$

Since the displacement is $y(t)$, we can use the fact that

- displacement is a position
- velocity is the first derivative of position
- acceleration is the first derivative of velocity

$$\therefore \quad v = \frac{dy}{dt}$$

$$a = \frac{dv}{dt} = \frac{d}{dt}\left[\frac{dy}{dt}\right] = \frac{d^2y}{dt^2}$$

$$\therefore \quad a = \frac{d^2y}{dt^2}$$

This means we can rewrite Newton's law as

$$F = m\frac{d^2y}{dt^2}$$

To complete the model, we must specify an expression for the force that the spring exerts on the mass. We use <u>Hooke's Law of Springs</u> as our model for the restoring force of the spring.

"The restoring force exerted by a spring is linearly proportional to the spring's displacement from its rest position and is directed toward the rest position."

$F_s = -ky$, where $k>0$ is called the spring constant

Now, by combining the two equations we have defined for force, we can obtain an equation for the motion of a mass attached to a spring.

$$F = -ky = m\frac{d^2y}{dt^2}$$

$$-ky = m\frac{d^2y}{dt^2}$$

$$m\frac{d^2y}{dt^2} + ky = 0$$

$$\boxed{\frac{d^2y}{dt^2} + \frac{k}{m}y = 0}$$

This is the equation for what scientists call a "simple" (or "undamped") harmonic oscillator.

Now, at first glance this may not appear to have anything in common with what we've done so far with systems of differential equations. However, even though this is a second order differential equation, we can write it as a system of first order differential equations.

Recall that

$$\frac{d^2y}{dt^2} = -\frac{k}{m}y$$

and

$$\frac{dv}{dt} = -\frac{k}{m}y$$

In other words

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -\frac{k}{m}y$$

Thus we can reduce the order of the differential equation (2) to a (1) by rewritting things as a combination of two first order differential equations.

Let's assume we have a spring and mass such that $\frac{k}{m} = 1$. This makes our original ODE

$$\frac{d^2y}{dt^2} = -y$$

which we can rewrite as

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -y$$

Now, let's stop for a minute and think about the motion of a mass attached to a spring. It is going to oscillate about the relaxation point (which we've already called $y=0$). The farthest away from $y=0$ is going to occur on both the positive (stretched) side of the spring and the negative (compressed) side. One it reaches this max/min point it returs the other direction. Is their a kind of function(s) that mimics this behavior?

or, think about it this way....

Is there a function(s) that is the negative of its second derivative?

$$y'' = -y$$

It turns out that there is a class of functions that do this... sines and cosines.

$y = \sin$	$y = \cos$
$y' = \cos$	$y' = -\sin$
$y'' = -\sin$	$y'' = -\cos$

Just by understand the behaviour of the spring, we already know that our solution must involve either a sine, a cosine, or both.

$$y = \sin \qquad \frac{dy}{dt} = v \qquad \Rightarrow \qquad \frac{d}{dt}[\sin] = \cos$$

$$\frac{dv}{dt} = -y \qquad \frac{d}{dt}[\cos] = -\sin$$

THE DAMPED HARMONIC OSCILLATOR

In the absence of friction, air resistance, or any other "damping" force, the harmonic oscillator will continue on forever. We know based on experience, this isn't the case in real-life. Thus, to improve upon the simple harmonic oscillator, we need to include a "damping term".

As a first guess model, we'll lump all the damping forces into a single term.

$$-b\left(\frac{dy}{dt}\right)$$

where $b > 0$ and is called the coefficient of damping. The minus sign indicates that the damping pushes against the direction of motion, always reducing the speed. Thus, our new model is now:

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt}$$

which is typically written as

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

This equation is often called the equation of motion of a "damped harmonic oscillator". We can simplify it further by dividing out the mass.

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$

where $p = \frac{b}{m}$

$q = \frac{k}{m}$

We can convert this new second-order differential equation into a system of first-order differential equations as follows

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -qy - pv$$

To get an idea of the behavior of ~~this~~ the solution of a damped harmonic oscillator, it would be nice to have some explicit solutions, and we can use "guess and check" method to get some.

Consider $$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$$

A solution $y(t)$ is a function whose second derivative can be expressed in terms of y, $\frac{dy}{dt}$, and constants.

The most familiar function whose derivative is almost exactly itself is the exponential function. So, let's guess a solution of the form

$$y(t) = e^{st} \qquad \text{where } s \text{ is any constant}$$

Plug this into our equation.

$$\begin{aligned} \frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y &= \frac{d^2}{dt^2}\left[e^{st}\right] + 5\frac{d}{dt}\left[e^{st}\right] + 6e^{st} \\ &= s^2e^{st} + 5se^{st} + 6e^{st} \\ &= (s^2 + 5s + 6)e^{st} \end{aligned}$$

In order for e^{st} to be a solution, this expression must equal the right-hand side of the original differential equation for all t. In other words

$$(s^2 + 5s + 6)e^{st} = 0$$

Now, $e^{st} \neq 0$ for all t, so that means (s^2+5s+6) must.

$$s^2 + 5s + 6 = 0$$

Use the quadratic equation to solve!

the above, only equals 0 when $s = -2$ or $s = -3$. This yields two possible solutions $Y_1(t) = e^{-2t}$ and $Y_2(t) = e^{-3t}$.

Now, we need to convert these two equations into solutions of our system.

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -5v + 6y$$

recall that we just solved for Y. V is simply the first derivative of y.

$$\therefore \quad Y_1(t) = e^{-2t}$$
$$V_1(t) = \frac{d}{dt}\left[e^{-2t}\right] = -2e^{-2t}$$

$$Y_2(t) = e^{-3t}$$
$$V_2(t) = \frac{d}{dt}\left[e^{-3t}\right] = -3e^{-3t}$$

Thus we now have $Y(t)$ and $V(t)$. This means we have two solutions to this system

$$\boxed{\vec{Y}_1(t) = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix} \qquad \vec{Y}_2(t) = \begin{bmatrix} e^{-3t} \\ -3e^{-3t} \end{bmatrix}}$$

EULER'S METHOD FOR AUTONOMOUS SYSTEMS

Euler's method for a system can be written as follows:

GIVEN: $\frac{dx}{dt} = f(x,y)$

$$\frac{dy}{dt} = g(x,y)$$

the initial condition (x_0, y_0), and the step-size Δt

$$m_k = f(x_k, y_k)$$

$$n_k = g(x_k, y_k)$$

$$x_{k+1} = x_k + m_k \Delta t$$

$$y_{k+1} = y_k + n_k \Delta t$$