PROPERTIES OF LINEAR SYSTEMS AND THE LINEARITY PRINCIPLE

Last time we discussed Harmonic Oscillators, but they won't represent all behaviors we need to study. So the generic linear system can be written as

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

where a, b, c, d are constants

This kind of system is called a "Linear System with constant Coefficients."

The most important adjective – linea – refers to the fact that the equations for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ involve only the first power of the dependent variable.

Because a, b, c, d are simply constants, this system is autonomous.

These systems have two dependent variables and are called "planar" or "two-dimensional". Since "two-dimensional, linear systems, with constant coefficients" is quite long, these systems are often simply called "planar linear systems" or "linear systems".

NOTE: These systems are two-variable generalizations of the homogeneous, constant-coefficient, first-order linear equation $\frac{dy}{dt} = ay$.

We can use vector and matrix notation to rewrite these systems.

Let $\vec{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (a 2×2 square matrix)

Let $\vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$ (a column vector)

Thus

$$\vec{A}\vec{Y} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

For Example

$$\begin{bmatrix} (2-a) & \pi \\ e & y \end{bmatrix} \begin{bmatrix} y \\ 2v \end{bmatrix} = \begin{bmatrix} (2-a)y + 2\pi v \\ ey + 2yv \end{bmatrix}$$

With this in mind, if x and y are the dependent variables, then

$$\vec{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \text{and} \quad \frac{d\vec{Y}(t)}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$$

Combining all of this, we can rewrite the general system in matrix notation

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned} \Longrightarrow \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} \Longrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

OR

$$\frac{d\vec{Y}}{dt} = \vec{A}\,\vec{Y}$$

where

$$\vec{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \vec{Y} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The matrix $\vec{A}$ of coefficients of the system is called the "coefficient matrix" one advantage of matrix and vector notation is that we can extend things to include any number of dependent variables.

$$y_1, y_2, y_3, \ldots, y_n$$

$$\frac{dy_1}{dt} = a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n$$

$$\frac{dy_2}{dt} = a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{dy_n}{dt} = a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{nn}y_n$$

In this case, the coefficients of this system are $a_{11}, a_{12}, \cdots a_{nn}$.

Let $\vec{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \Longrightarrow \frac{d\vec{Y}}{dt} = \begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \vdots \\ \frac{dy_n}{dt} \end{pmatrix}$

The coefficient Matrix is the $n \times n$ matrix

$$\vec{A} = \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}$$

and we have

$$\frac{d\vec{Y}}{dt} = \vec{A}\vec{Y} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n \\ a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n \\ \vdots \qquad \vdots \qquad \vdots \\ a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{nn}y_n \end{bmatrix}$$

The number of dependent variables is called the "dimension". In the above case we have n-dependent variables so we have an n-dimensional system.

For example, the three-dimensional system

$$\frac{dx}{dt} = \sqrt{2}\,x + y$$

$$\frac{dy}{dt} = z$$

$$\frac{dz}{dt} = -x - y + 2z$$

can be written as

$$\frac{d\vec{Y}}{dt} = \vec{A}\,\vec{Y}$$

where $\vec{Y} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\vec{A} = \begin{bmatrix} \sqrt{2} & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 2 \end{bmatrix}$

Equilibrium Points of Linear Systems and the Determinant

We start our "solving" journey by examining the simplest of solutions — the equilibrium solutions. Recall that

$$\vec{Y}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

is an equilibrium point of a system if and only if the vector field at $\vec{Y}_0$ is the zero vector. Since the vector field of a system at the point $\vec{Y}_0$ is given by the right hand side of the differential equation evaluated at that point and since $\frac{d\vec{Y}}{dt} = \vec{A}\vec{Y}$ for a linear system, we know that the vector field

$\vec{F}(\vec{Y}_0)$ at $\vec{Y}_0$ for a linear system is given by

$$\vec{F}(\vec{Y}_0) = \vec{A}\,\vec{Y}_0$$

In other words, the vector at $\vec{Y}_0$ is computed by taking the product of the matrix $\vec{A}$ with the vector $\vec{Y}_0$.

Based on this, the equilibrium points are the points $\vec{Y}_0$ where

$$\vec{A}\vec{Y}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

That is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} ax_0 + by_0 \\ cx_0 + dy_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Written in scalar form, we get the pair of simultaneous linear equations

$$ax_0 + by_0 = 0$$
$$cx_0 + dy_0 = 0$$

Clearly $(x_0, y_0) = (0,0)$ is a solution to these equations. Therefore, the point $\vec{Y}_0 = (0,0)$ is an equilibrium point, and the constant function

$$\vec{Y}(t) = (0,0) \quad \text{for all } t$$

is a solution to the linear system. This solution is known as the "TRIVIAL SOLUTION".

Any other equilibrium points (x_0, y_0) must also satisfy

$$ax_0 + by_0 = 0$$
$$cx_0 + dy_0 = 0$$

To find them, assume for a moment that $a \neq 0$. Using the first equation, we can solve for x_0.

$$x_0 = -\frac{b}{a} y_0$$

The second equation then becomes

$$c\left[-\frac{b}{a} y_0\right] + dy_0 = 0$$

multiply by "a"

$$-cby_0 + ady_0 = 0$$

$$\boxed{(ad - cb)\, y_0 = 0}$$

Hence, either $Y_0 = 0$ or $(ad-bc) = 0$. If $Y_0 = 0$, then $X_0 = 0$ and we once again have the trivial solution. Therefore, a linear system has non-trivial equilibrium points only if $ad - bc = 0$.

It turns out that $ad-bc$ is nothing more than the determinant of $\vec{A}$!

THEOREM

If $\vec{A}$ is a matrix with $\det \vec{A} \neq 0$, then the only equilibrium point for the system $\frac{d\vec{Y}}{dt} = \vec{A}\vec{Y}$ is the origin

EXAMPLE

$$\vec{A} = \begin{pmatrix} 2 & 1 \\ -4 & 0.3 \end{pmatrix}$$

$$\det \vec{A} = (2)(0.3) - (-4)(1)$$
$$0.6 + 4$$
$$\det \vec{A} = 4.6$$

Since $\det \vec{A} \neq 0$, the only equilibrium points for the system $\frac{d\vec{Y}}{dt} = \vec{A}\vec{Y}$ is the origin.

An Important Property of the Determinant

The determinant is a ~~pro~~ quantity that pops up a lot for the remainder of the course. Most of the time all we care about is if the determinant is 0. If we were to pick four numbers at random, then it is unlikely that ad-bc = 0. Thus, matrices whose determinants are 0 are often called singular or degenerate. From the previous theorem, we know that if a linear system $\frac{d\vec{Y}}{dt} = \vec{A}\vec{Y}$ is "non-degenerate" ($\det \vec{A} \neq 0$), then it has exactly one equilibrium point, (0,0). In other words, an initial condition of (0,0) corresponds to a solution curve that sits at (0,0) for all time. Any other initial condition yields a solution that changes with time.

It is important to understand this concept.

!!! What we actually proved is that a system of equations has "non-trivial solutions" [solutions other than (0,0)] if and only if $\det \vec{A} = 0$!!