The Linearity Principle

The solutions of linear systems have special properties that solutions of arbitrary systems do not have. These properties are so useful that we take advantage of them repeatedly. In fact they are exactly the reason that we will be so successful in our analysis of linear systems.

However, a note of caution: It is important to make sure that the system under consideration actually is a linear system before you use any of these special properties.

THE MOST IMPORTANT PROPERTY OF LINEAR SYSTEMS IS THE LINEARITY PRINCIPLE.

LINEARITY PRINCIPLE

Suppose $\frac{d\vec{Y}}{dt} = \vec{A}\vec{Y}$ is a linear system of differential equations

1. If $\vec{Y}(t)$ is a solution of this system and k is any constant, then $k\vec{Y}(t)$ is also a solution.

2. If $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ are solutions of the system, then $\vec{Y}_1(t) + \vec{Y}_2(t)$ is also a solution.

The LINEARITY PRINCIPLE is sometimes called THE PRINCIPLE OF SUPERPOSITION.

Combining the two parts of the Linearity Principle we see that

$$k_1\vec{Y}_1(t) + k_2\vec{Y}_2(t) \text{ is also a solution.}$$

A solution of ~~them~~ this form $\left(k_1\vec{Y}_1(t) + k_2\vec{Y}_2(t)\right)$ is called a linear combination of the solutions $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$

VERIFICATION OF THE LINEARITY PRINCIPLE

To show that the linearity principle holds in general, we first state the following two algebraic properties of matrix multiplication.

1. If $\vec{A}$ is a matrix and $\vec{Y}$ is a vector, then

$$\vec{A}(k\vec{Y}) = k\vec{A}\vec{Y}$$

for any constant k.

2. If $\vec{A}$ is a matrix and $\vec{Y}_1$ and $\vec{Y}_2$ are vectors, then

$$\vec{A}(\vec{Y}_1 + \vec{Y}_2) = A\vec{Y}_1 + A\vec{Y}_2$$

We can verify these two facts for 2x2 matrices and 2-dimensional vectors by direct computation. For example

$$\vec{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\vec{Y}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad \text{Then} \quad \vec{A}(\vec{Y}_1 + \vec{Y}_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

$$\vec{Y}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \qquad = \begin{bmatrix} a(x_1+x_2) + b(y_1+y_2) \\ c(x_1+x_2) + d(y_1+y_2) \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + ax_2 + by_1 + by_2 \\ cx_1 + cx_2 + dy_1 + dy_2 \end{bmatrix}$$

AND

$$A\vec{Y}_1 + A\vec{Y}_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$= \begin{bmatrix} ax_1 + by_1 \\ cx_1 + dx_1 \end{bmatrix} + \begin{bmatrix} ax_2 + by_2 \\ cx_2 + dy_2 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + ax_2 + by_1 + by_2 \\ cx_1 + cx_2 + dy_1 + dy_2 \end{bmatrix}$$

are equal

Because the previous two expansions are equal

$$A(\vec{Y}_1 + \vec{Y}_2) = A\vec{Y}_1 + A\vec{Y}_2$$

Now, given these algebraic properties of matrix multiplication, we can verify the linearity principle using the standard rules of differentiation. Now, suppose $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ are solutions to $\frac{d\vec{Y}(t)}{dt} = \vec{A}\vec{Y}$. That is

$$\frac{d\vec{Y}_1}{dt} = A\vec{Y}_1 \quad \text{and} \quad \frac{d\vec{Y}_2}{dt} = A\vec{Y}_2 \quad \text{for all } t$$

For any constant k we have

$$\frac{d(k\vec{Y}_1)}{dt} = k\frac{d\vec{Y}_1}{dt} = k\vec{A}\vec{Y}_1 = \vec{A}(k\vec{Y}_1)$$

so $k\vec{Y}_1(t)$ is a solution to the system. Also

$$\frac{d(\vec{Y}_1 + \vec{Y}_2)}{dt} = \frac{d\vec{Y}_1}{dt} + \frac{d\vec{Y}_2}{dt} = A\vec{Y}_1 + A\vec{Y}_2 = A(\vec{Y}_1 + \vec{Y}_2) \quad \text{for all } t.$$

Solving Initial Value Problems

From the linearity principle we know that, given two solutions $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ we can make many more solutions of the form $k_1\vec{Y}_1 + k_2\vec{Y}_2$ for any constants k_1 and k_2. This type of expression is called a "two-parameter family of solutions" since we have two constants (k_1 and k_2) that we can adjust to obtain various solutions.

Consider:

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix} \vec{Y}.$$

Suppose we want to find the solution $\vec{Y}(t)$ of this system with initial value
$\vec{Y}_0 = (2,-3)$.

We solved this problem in the last notes and the solutions were

$$Y_1(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} \quad \text{and} \quad Y_2(t) = \begin{pmatrix} -e^{-4t} \\ 2e^{-4t} \end{pmatrix}.$$

Now, if we plug in 0 for t we see that

$$Y_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad Y_2(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Now, wait. Neither of our solutions for Y when $t=0$ are equal to what we were told our initial condition ($Y_0 = (2,-3)$) actually was. So what do we do?

Turns out that the linearity Principle says we can form any linear combination of $\vec{Y}_1(t)$ and $Y_2(t)$ and still have a solution. Hence, we need values of K_1 and K_2, such that

$$K_1 \vec{Y}_1(0) + K_2 \vec{Y}_2(0) = \vec{Y}_0$$

so

$$K_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + K_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} K \\ 0 \end{pmatrix} + \begin{pmatrix} -K_2 \\ 2K_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$K - K_2 = 2$$
$$2K_2 = -3$$

thus $K_2 = -3/2$

$K_1 = 1/2$.

This means

$$\tfrac{1}{2}\vec{Y}_1(0) - \tfrac{3}{2}\vec{Y}_2(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

So our solution is then

$$\vec{Y}(t) = \tfrac{1}{2}\vec{Y}_1(t) - \tfrac{3}{2}\vec{Y}_2(t)$$

$$= \tfrac{1}{2}\begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} - \tfrac{3}{2}\begin{pmatrix} -e^{-4t} \\ 2e^{-4t} \end{pmatrix}$$

$$\vec{Y}(t) = \begin{bmatrix} \tfrac{1}{2}e^{2t} + \tfrac{3}{2}e^{-4t} \\ -3e^{-4t} \end{bmatrix} \longleftarrow$$

Specific (Particular) Solution to the given IVP.

What is the general solution?

Go back to

$$\vec{Y}_1(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{Y}_2(t) = \begin{pmatrix} -e^{-4t} \\ 2e^{-4t} \end{pmatrix}$$

the general solution is simply

$$\vec{Y}(t) = k_1\vec{Y}_1(t) + k_2\vec{Y}_2(t)$$

$$= k_1\begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + k_2\begin{pmatrix} -e^{-4t} \\ 2e^{-4t} \end{pmatrix}$$

$$= \begin{pmatrix} k_1e^{2t} \\ 0 \end{pmatrix} + \begin{pmatrix} -k_2e^{-4t} \\ 2k_2e^{-4t} \end{pmatrix}$$

$$\boxed{\vec{Y}(t) = \begin{pmatrix} k_1e^{2t} - k_2e^{-4t} \\ 2k_2e^{-4t} \end{pmatrix}}$$

LINEAR INDEPENDENCE

Expressing arbitrary vectors as linear combinations of given vectors is a fundamental topic in linear algebra. In the two-dimensional case, the key property that ensures that an arbitrary vector can be written as some linear combination of the given vectors (x_1, y_1) and (x_2, y_2) is that they do not lie on the same line through the origin.

We say that these two vectors are

LINEARLY INDEPENDENT

if they do not lie on the same line through the origin. Another way to say this is that neither vector is a multiple of the other.

THEOREM

Suppose (x_1, y_1) and (x_2, y_2) are two linearly independent vectors in the plane. Then given any vector (x_0, y_0), there exist k_1 and k_2 such that

$$k_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + k_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Two linearly independent vectors can be combined via addition and scalar multiplication to form any other vector in the plane.

THE GENERAL SOLUTION

THEOREM: Suppose $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ are solutions of the linear system

$$\frac{d\vec{Y}}{dt} = \vec{A}\vec{Y},$$

If $\vec{Y}_1(0)$ and $\vec{Y}_2(0)$ are linearly independent, then for any initial condition

$\vec{Y}(0) = (x_0, y_0)$, we can find constants k_1 and k_2 so that

$k_1 \vec{Y}_1(t) + k_2 \vec{Y}_2(t)$ is the solution to the initial value problem

$$\frac{d\vec{Y}}{dt} = \vec{A}\vec{Y} \;;\; \vec{Y}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

In this case

$\vec{Y}(t) = k_1 \vec{Y}_1(t) + k_2 \vec{Y}_2(t)$ is the general solution!

An Undamped (Simple) Harmonic Oscillator

Recall the equation for a simple harmonic oscillator is

$$\frac{d^2y}{dt^2} = -y$$

which can be rewritten as a linear system

$$\frac{dy}{dt} = v \qquad \text{or} \qquad \vec{Y}(t) = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix}$$

$$\frac{dv}{dt} = -y$$

We can rewrite the system in matrix notation like this

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{Y}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}$$

Recall that when we previously solved this problem we guessed the solution would consist of either sines or cosines. For this exercise let's assume $y_1 = \cos t$

Therefore $\vec{Y}(t) = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$

$\frac{d\vec{Y}}{dt}$ becomes $\frac{d\vec{Y}}{dt} = \begin{bmatrix} \frac{dy}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} \frac{d}{dt} \cos t \\ \frac{d}{dt} -\sin t \end{bmatrix} = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}$

and multiplied out

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}$$

so $Y_1 = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$

Now let $y_2 = \sin t$.

Going through the same excercises as last page we find

$$\vec{Y}_2 = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

Now, at this point we have a first order linear system (the simple harmonic oscillator) with two dependent variables. This means we need two linearly independent solutions to obtain the general solution.

Well, at $t=0$

$$\vec{Y}_1(0) = \begin{bmatrix} \cos(0) \\ -\sin(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and $$\vec{Y}_2(0) = \begin{bmatrix} \sin(0) \\ \cos(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This means that $\vec{Y}_1$ lies on the y-axis and $\vec{Y}_2$ lies on the v-axis. Thus, because these two solutions are on separate axes, they cannot be multiples of each other. This means we can use the Linearity Principle to combine them for the general solution.

[illegible] = [illegible]

$$\boxed{Y(t) = k_1\vec{Y}_1(t) + k_2\vec{Y}_2(t)}$$

So the ~~[illegible]~~ general solution becomes

$$\boxed{Y(t) = k_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + k_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}}$$