EXAM #1

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1.

1.) A differential equation is ~~a~~ an equation involving an unknown function of at least 1 variable and its derivative

2.) A first order differential equation is one where the ~~highest~~ order of the derivative is 1. In other words, the differential equation ~~is~~ contains only first derivatives, and no derivatives of higher orders.

3.) A differential equation is separable if the function $f(t,y)$ can be written as the product of two functions: one that depends solely on t and one that depends solely on y.

4.) A differential equation is said to be autonomous if the independent variable is equal to 0.

5.) A linear differential equation is one where there are no multiplications among dependent variables and their derivatives. In other words, all coefficients are functions of independent variables.

6.) ~~A~~ The slope of the solution is constant along the ~~x~~-axis.

7.) A homogeneous differential equation has the ~~a~~ ~~simple~~ form:

$$\frac{dy}{dt} = a(t)y$$

where as a nonhomogeneous ODE has the following form

$$\frac{dy}{dt} = a(t)y + b(t)$$

8.) $\frac{dy}{dt} = y$

9.) $\frac{dy}{dt} = y$

$$\frac{dy}{y} = dt$$

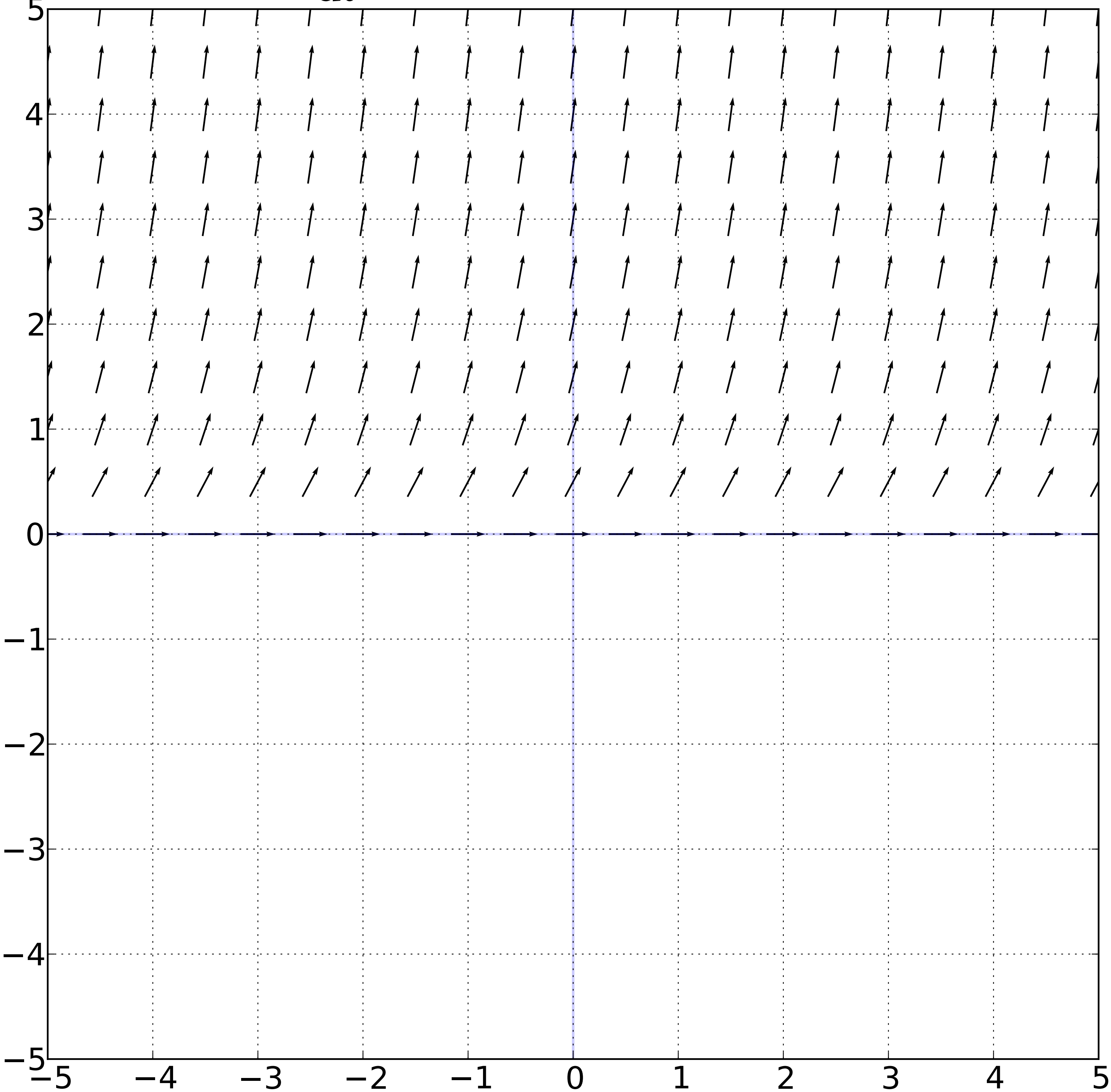
$$\ln y = t + c$$

$$e^{\ln y} = e^{t+c}$$

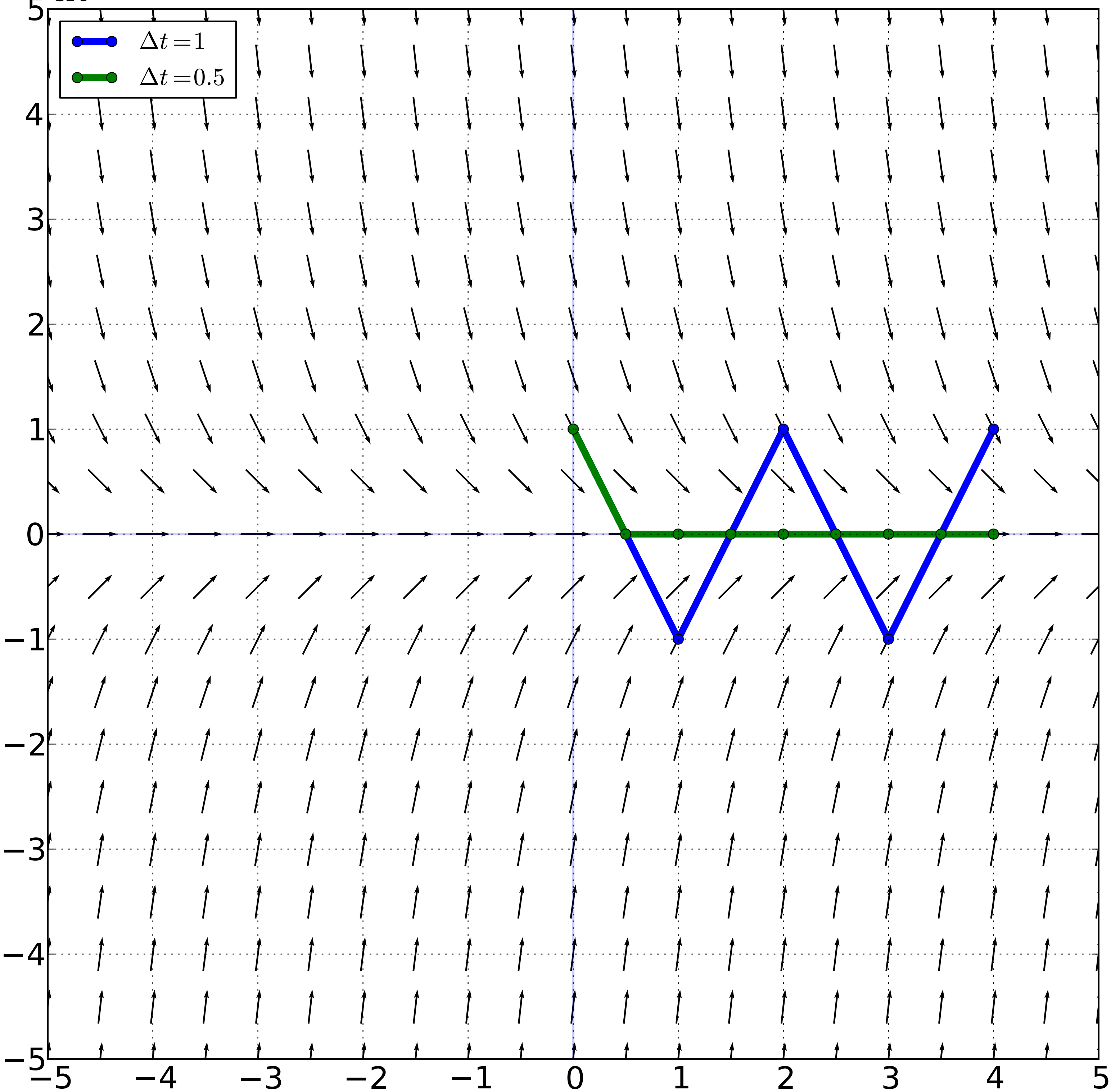
$$\boxed{y(t) = Ce^{t}}$$

10.) No!

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3*y**(2/3)$$



$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2 * y + 2 * np.sin(2 * np.pi * t)$$



4.]

(1a) linear, homogeneous

(1b.) $\frac{dy}{dt} = -2ty \quad y(0) = e$

$\frac{dy}{y} = -2t\,dt$

$\ln y = -t^2 + C$

$e^{\ln y} = e^{-t^2 + C}$

$\boxed{y(t) = Ce^{-t^2}}$

$y(0) = e$

$e = Ce^0$

$e = C$

$\boxed{y(t) = ee^{-t^2} \text{ or } y(t) = e^{1-t^2}}$

(2a.) linear

$\frac{dy}{dt} = 2y + \cos 4t \quad y(0) = 1$

$\frac{dy}{dt} = 2y$

$\frac{dy}{y} = 2dt$

$\ln y = 2t + C$

$e^{\ln y} = e^{2t + C}$

$\boxed{y(t) = Ce^{2t}}$

GUESS: $\alpha \cos 4t + \beta \sin 4t$

$\frac{dy}{dt} - 2y = \cos 4t$

$\frac{d}{dt}(\alpha \cos 4t + \beta \sin 4t) - 2(\alpha \cos 4t + \beta \sin 4t) = \cos$

$-4\alpha \sin 4t + 4\beta \cos 4t - 2\alpha \cos 4t - 2\beta \sin 4t = \cos$

want to keep $\boxed{(-2\alpha + 4\beta)\cos 4t} + (-4\alpha - 2\beta)\sin 4t = \cos 4$

$-2\alpha + 4\beta = 1$

$2\cdot(-4\alpha - 2\beta = 0)$

$-2\alpha + 4\beta = 1$

$-8\alpha - 4\beta = 0$

$-10\alpha = 1$

$\boxed{\alpha = -\frac{1}{10}}$

$-2(-\frac{1}{10}) + 4\beta = 1$

$\frac{1}{5} + 4\beta = 1$

$4\beta = \frac{4}{5}$

$\boxed{\beta = \frac{1}{5}}$

$\boxed{y(t) = Ce^{2t} - \frac{1}{10}\cos 4t + \frac{1}{5}\sin 4t}$

$y(0) = 1$

$1 = Ce^0 - \frac{1}{10}\cos(0) + \frac{1}{5}\sin(0)$

$1 = C - \frac{1}{10}(1)$

$\boxed{\frac{11}{10} = C}$

$\boxed{y(t) = \frac{11}{10}e^{2t} - \frac{1}{10}\cos 4t + \frac{1}{5}\sin 4t}$

(3a.) homogeneous

(3b.) $\frac{dy}{dt} = t^2 y^3 + y^3$ $\qquad y(0) = -\frac{1}{2}$

$$\frac{dy}{dt} = (t^2+1)y^3$$

$$\frac{dy}{y^3} = (t^2+1)dt$$

$$y^{-3}dy = \int t^2 dt + \int dt$$

$$-\frac{1}{2}y^{-2} = \frac{t^3}{3} + t + C$$

$$y^{-2} = -\frac{2}{3}t^3 - 2t + C$$

$$y^2 = \frac{1}{-\frac{2}{3}t^3 - 2t + C}$$

$y(0) = \frac{1}{2}$

$$\left(\frac{1}{2}\right)^2 = \frac{1}{-\frac{2}{3}(0) - 2(0) + C}$$

$$\frac{1}{4} = \frac{1}{C}$$

$$\boxed{C=4}$$

$$y^2 = \frac{1}{-\frac{2}{3}t^3 - 2t + 4}$$

$$y = \frac{\pm 1}{\sqrt{-\frac{2}{3}t^3 - 2t + 4}}$$

we take -1 since initial condition was negative

$$\boxed{y(t) = \frac{-1}{\sqrt{-\frac{2}{3}t^3 - 2t + 4}} \quad \text{or} \quad y(t) = \frac{-1}{\sqrt{4 - 2t - \frac{2}{3}t^3}}}$$

4] (4a.) linear ~~[illegible]~~

(4b.) $\frac{dy}{dt} = 2ty + 3te^{t^2}$ $y(0) = 1$

$\frac{dy}{dt} - 2ty = 3te^{t^2}$

$\mu(t) = e^{\int g(t)\,dt} = e^{-\int 2t\,dt} = e^{-t^2}$

$\boxed{\mu(t) = e^{-t^2}}$

$e^{-t^2}\left[\frac{dy}{dt} - 2ty\right] = 3te^{t^2}$

$e^{-t^2}\frac{dy}{dt} - e^{-t^2}2ty = 3te^{t^2}e^{-t^2}$

$\frac{d}{dt}\left[e^{-t^2}y\right] = 3t$

$d\left[e^{-t^2}y\right] = 3t\,dt$

$e^{-t^2}y = \frac{3}{2}t^2 + C$

$\boxed{y = \left[\frac{3}{2}t^2 + C\right]e^{t^2}}$

$1 = \left[\frac{3}{2}(0) + C\right]e^0$

$\boxed{C = 1}$

$\cancel{y(t) = \frac{3}{2}t^2 e^{t^2}}$

$\boxed{y(t) = \left[\frac{3}{2}t^2 + 1\right]e^{t^2}}$

Q4 (5a) autonomous

(5b.) $\frac{dy}{dt} = 1 - y^2 \qquad y(0) = 1$

we can see that the initial value y(0)=1 holds for all t. Therefore the solution is the equilibrium solution

$$\boxed{y(t) = 1}$$

BONUS

(a.) Let C(t) be the volume of carbon monoxide at time t, where t is measured in hours. Initially, the amount of the carbon monoxide is 3% by volume. Since the volume of the room is 1000 cubic feet, there are 30 cubic feet of carbon monoxide in the room at time t=0. Carbon monoxide is being blown into the room at a rate of 1 cubic foot per hour. The concetration of carbon monoxide is $\frac{C}{1000}$, so carbon monoxide leaves the room at a rate of

$$100\left(\frac{C}{1000}\right).$$

The initial value problem that models this situation is

$$\boxed{\frac{dC}{dt} = 1 - \frac{C}{10} \ ; \quad C(0) = 30}$$

(b.) $\frac{dC}{dt} = 1 - \frac{C}{10}$

$\frac{dC}{dt} = -\frac{C}{10}$

$\frac{dC}{C} = -\frac{1}{10}\,dt$

$\ln C = -\frac{t}{10} + K$

$e^{\ln C} = e^{-\frac{t}{10} + K}$

$\boxed{C = Ke^{-0.1t}}$

Homogeneous solution

As with any autonomous differential equations an equilibrium solution occurs any time the right hand side is equal to 0. To use the Extendend Linearity Principle, we only need to know one particular solution. therefore, we can use the equilibrium solution as our particular solution.

$1 - \frac{C}{10} = 0$

$1 = \frac{C}{10}$

$\boxed{10 = C}$

$\therefore \quad \boxed{C(t) = 10 + Ke^{-0.1t}}$

BONUS

(b cont.)

$$C(t) = 10 + Ke^{-0.1t} \qquad C(0) = 30$$

$$30 = 10 + Ke^{(0)}$$

$$30 = 10 + K$$

$$\boxed{20 = K}$$

$$\therefore \boxed{C(t) = 10 + 20e^{-0.1t}}$$

Now, we want to know t when $C(t) = 2\%$. Well 2% is equal to 20 cubic feet. therefore we want to solve for t when $C(t) = 20$

$$\therefore \quad 20 = 10 + 20e^{-0.1(t)}$$

$$10 = 20e^{-0.1t}$$

$$0.5 = e^{-0.1t}$$

$$\ln(0.5) = -0.1t$$

$$t = -10 \cdot \ln(0.5)$$

$$t = -10 \cdot -0.693$$

$$\boxed{\begin{array}{l} t = 6.9 \text{ hours} \\ \text{OR} \\ t = 6 \text{ hours and } 54 \text{ minutes} \end{array}}$$