L

(1a.) $\frac{dy}{dt} = ty$

$\int \frac{dy}{y} = \int t\, dt$

$\ln|y| = \frac{1}{2}t^2 + c$

$e^{\ln|y|} = e^{\frac{1}{2}t^2 + c}$

$\boxed{Y(t) = Ce^{t^2/2}}$

(1b.) $\frac{dy}{dt} = t^4 y$

$\int \frac{dy}{y} = \int t^4\, dt$

$\ln|y| = \frac{1}{5}t^5 + C$

$e^{\ln|y|} = e^{\frac{1}{5}t^5 + C}$

$\boxed{Y(t) = Ce^{t^5/5}}$

(1c) $\frac{dy}{dt} = e^{-y}$

$\frac{dy}{e^{-y}} = dt$

$\int e^y dy = \int dt$

$e^y = t + c$

$\ln|e^y| = \ln|t+c|$

$\boxed{Y(t) = \ln|t+c|}$

(1d) $\frac{dx}{dt} = 1 + x^2$

$\int \frac{dx}{1+x^2} = \int dt$

$\arctan(x) = t + c$

$\tan[\arctan(x)] = \tan(t+c)$

$\boxed{X(t) = \tan(t+c)}$

(1e) $\frac{dy}{dt} = \frac{t}{t^2 y + y}$

$\frac{dy}{dt} = \frac{t}{y(t^2+1)}$

$\int y\, dy = \int \frac{t}{t^2+1}\, dt$

$\frac{1}{2}y^2 = \int \frac{t}{t^2+1} \cdot 2 \cdot \frac{1}{2} \cdot dt$

$\frac{1}{2}y^2 = \frac{1}{2}\int \frac{2t}{t^2+1}\, dt$

$\frac{1}{2}y^2 = \frac{1}{2}\ln|t^2+1| + c$

$y^2 = \ln|c[t^2+1]|$

$\boxed{Y(t) = \sqrt{\ln|c[t^2+1]|}}$

(1f) $\frac{dy}{dt} = t \sqrt[3]{y}$

$\frac{dy}{dt} = t \cdot y^{\frac{1}{3}}$

$\frac{dy}{y^{1/3}} = t\, dt$

$\int y^{-1/3} dy = \int t \cdot dt$

$\frac{3}{2} y^{2/3} = \frac{1}{2}t^2 + C_1$

$y^{2/3} = \frac{t^2}{3} + C$

$\boxed{Y(t) = \pm \left[\frac{t^2}{3} + C\right]^{3/2}}$

NOTE. THIS DOES NOT CONTAIN $Y(t) = 0$!

(2a.) $\frac{dy}{dt} = -y^2$; $y(0)=0$

$$-\frac{dy}{y^2} = dt$$

$$\frac{1}{y} = t + C$$

$$\boxed{Y(t) = \frac{1}{t+c}}$$

Now, the only way to get

$y(0)=0$ is ~~for~~ to

use the trivial solution!

$$y(0) = 0$$

$$\boxed{\therefore \; y(t) = 0}$$

(2b.) $\frac{dy}{dt} = -y^2$; $y(0) = \frac{1}{2}$

From last time:

$$\boxed{Y(t) = \frac{1}{t+c}}$$

solving for c

$$\frac{1}{2} = \frac{1}{0+c}$$

$$\frac{1}{2} = \frac{1}{c}$$

$$c = 2$$

$$\therefore \; \boxed{Y(t) = \frac{1}{t+2}}$$

(2c.) $\frac{dy}{dt} = 2y+1$; $y(0) = 3$

$$\int \frac{dy}{2y+1} = \int dt$$

$$\int \frac{dy}{2y+1} \cdot \frac{1}{2} \cdot 2 = \int dt$$

$$\frac{1}{2} \int \frac{2dy}{2y+1} = \int dt$$

$$\frac{1}{2} \ln|2y+1| = t + C_1$$

$$\ln|2y+1| = 2t + C_2$$

$$e^{\ln|2y+1|} = e^{2t+C_2}$$

$$|2y+1| = C_3 e^{2t}$$

$$2y = C_3 e^{2t} - 1$$

$$\boxed{Y(t) = Ce^{2t} - \frac{1}{2}}$$

$$3 = Ce^{2(0)} - \frac{1}{2}$$

$$3 = C - \frac{1}{2}$$

$$\frac{7}{2} = C$$

$$\therefore \; \boxed{Y(t) = \frac{7}{2} e^{2t} - \frac{1}{2}}$$

(2d.) $\frac{dy}{dt} = t^2 y^3$; $y(0) = -1$

$$\frac{dy}{y^3} = t^2 dt$$

$$\int y^{-3} dy = \int t^2 dt$$

$$-\frac{1}{2} y^{-2} = \frac{1}{3}t^3 + C_1$$

$$y^{-2} = -\frac{2}{3}t^3 + C$$

$$y^2 = \frac{1}{-\frac{2}{3}t^3 + C}$$

$$\boxed{Y(t) = \pm \sqrt{\frac{1}{C - \frac{2}{3}t^3}}}$$

Since $y(0) = -1$ is negative, choose negative root.

$$-1 = -\sqrt{\frac{1}{C - \frac{2}{3}0^3}}$$

$$-1 = -\sqrt{\frac{1}{C-0}}$$

$$-1 = -\sqrt{\frac{1}{C}}$$

$$1 = \sqrt{\frac{1}{C}}$$

$$1 = \frac{1}{C}$$

$$C = 1$$

$$\boxed{Y(t) = -\sqrt{\frac{1}{1 - \frac{2}{3}t^3}}}$$

3

(2e) $\frac{dy}{dt} = \frac{t}{y - t^2 y}$; $y(0) = 4$

$$\frac{dy}{dt} = \frac{t}{y(1-t^2)}$$

$$\int y\,dy = \int \frac{t}{1-t^2}\,dt$$

$$\frac{1}{2}y^2 = \int \frac{t}{1-t^2} \cdot -\frac{1}{2} \cdot -2 \cdot dt$$

$$\frac{1}{2}y^2 = -\frac{1}{2}\int \frac{-2t}{1-t^2}\,dt$$

$$\frac{1}{2}y^2 = -\frac{1}{2}\ln|1-t^2| + c$$

$$y^2 = -\ln|1-t^2| + c$$

$$\boxed{y(t) = \pm\sqrt{-\ln|1-t^2| + c}}$$

Since $y(0)$ = positive, use positive root

$$4 = \pm\sqrt{-\ln|1-0^2| + c}$$

$$4 = +\sqrt{-\ln|1| + c}$$

$$4 = +\sqrt{0 + c}$$

$$4 = +\sqrt{c}$$

$$16 = c$$

$$y(t) = +\sqrt{-\ln|1-t^2| + 16}$$

OR

$$y(t) = +\sqrt{16 - \ln|1-t^2|}$$

(2f) $\frac{dx}{dt} = \frac{t^2}{x + t^3 x}$; $x(0) = -2$

$$\frac{dx}{dt} = \frac{t^2}{x(1+t^3)}$$

$$\int x\,dx = \int \frac{t^2}{1+t^3}\,dt$$

$$\frac{1}{2}x^2 = \int \frac{t^2}{1+t^3} \cdot \frac{1}{3} \cdot 3 \cdot dt$$

$$\frac{1}{2}x^2 = \frac{1}{3}\int \frac{3t^2\,dt}{1+t^3}$$

$$\frac{1}{2}x^2 = \frac{1}{3}\ln|1+t^3| + C_1$$

$$x^2 = \frac{2}{3}\ln|1+t^3| + C$$

$$\boxed{x = \pm\sqrt{\frac{2}{3}\ln|1+t^3| + C}}$$

Since $x(0)$ is negative, use negative root

$$-2 = -\sqrt{\frac{2}{3}\ln|1+0^3| + C}$$

$$2 = \sqrt{\frac{2}{3}\ln|1+0^3| + C}$$

$$4 = \frac{2}{3}\ln|1| + C$$

$$4 = \frac{2}{3} \cdot 0 + C$$

$$4 = C$$

$$\therefore \boxed{x(t) = -\sqrt{\frac{2}{3}\ln|1+t^3| + 4}}$$

(BONUS) $\frac{dA}{dt} = 0.02A$

$$\int \frac{dA}{A} = \int 0.02\, dt$$

$$\ln|A| = 0.02t + C$$

$$e^{\ln|A|} = e^{0.02t + C}$$

$$A = Ce^{0.02t}$$

$$C = A(0) = \$5{,}000$$

$\therefore \quad A(t) = 5000e^{0.02t}$

after 10 years $(t=10)$

$$A(10) = 5000e^{0.02 \cdot 10}$$

$$A(10) = 5000e^{0.2}$$

$$A(10) = 5000 \cdot 1.221$$

$$\boxed{A(10) = 6107.01}$$

We actually have a piece-wise ODE

$$\frac{dA}{dt} = \begin{cases} 0.02A & \text{for } t<10 \\ 0.02A - 500 & \text{for } t>10 \end{cases}$$

To solve this 2-part ODE we solve the first piece for $A(10)$ which we just did. We then use this as the initial condition for the second problem. $\therefore$

$$\frac{dA}{dt} = 0.02A - 500 \quad ; \quad A(10) = 6107.01$$

$$\int \frac{dA}{0.02A - 500} = \int dt$$

We have to use u-substitution...

Let $u = 0.02A - 500$. then

$du = 0.02\,dA$ or $50\,du = dA$

$\therefore \quad \int \frac{50\,du}{u} = \int dt$

$$\int \frac{50\,du}{u} = \int dt$$

$$50 \int \frac{du}{u} = \int dt$$

$$50 \ln|u| = t + c$$

$$50 \cdot \ln|0.02A - 500| = t + C$$

At $t=10$, we know $A = 6107.01$ thus

$$\frac{dA}{dt} = 0.02(6107.01) - 500$$

$$= 122.14 - 500$$

$$= -377.86$$

Since $\frac{dA}{dt}$ is negative, this means we are taking out more money than we put in... meaning it will run out.

If $0.02A - 500 < 0$, then

$$|0.02A - 500| = -(0.02A - 500) = \boxed{500 - 0.02A}$$

$\therefore \quad 50 \cdot \ln(500 - 0.02A) = t + C_1$

$$\ln(500 - 0.02A) = \tfrac{1}{50}t + C_1$$

$$500 - 0.02A = e^{\frac{1}{50}t + C_1}$$

$$0.02A = 500 - C_2 e^{\frac{1}{50}t}$$

$$A = 50\left[500 - C_2 e^{\frac{1}{50}t}\right]$$

Solve for C_3 using $A(10) = 6107.01$

$$A = 25000 - C_3 e^{\frac{1}{50}t}$$

$$6107.01 = 25000 - C_3 e^{\frac{1}{50} \cdot 10}$$

$$C_3 e^{\frac{1}{50} \cdot 10} = 25000 - 6107.01$$

$$C_3 = \frac{25000 - 6107.01}{e^{1/5}}$$

$$\boxed{C_3 = 15468.27}$$

So, $C_3 = 15468.27$

Now, we need to solve for when we run out of money.

$A(t) = 0$ ... solve for t

$$0 = A(t) = 25000 - 15468.27 \cdot e^{\frac{1}{50} \cdot t}$$

$$15468.27\, e^{\frac{1}{50}t} = 25000$$

$$e^{\frac{1}{50}t} = \frac{25000}{15468.27}$$

$$\ln\left|e^{\frac{1}{50}t}\right| = \ln\left|\frac{25000}{15468.27}\right|$$

$$\frac{1}{50}t = \ln\left|\frac{25000}{15468.27}\right|$$

$$t = 50 \cdot \ln\left|\frac{25000}{15468.27}\right|$$

$$t = 50 \cdot \ln|1.616|$$

$$t = 50 \cdot 0.48$$

$$\boxed{t = 24.00 \text{ years}}$$

Money will run out in slightly over 24 years.
This means there will be 14 years of withdraws