(1i) $\frac{dy}{dt} = y + e^{-2t}$

associated homogeneous

$\frac{dy}{dt} = y$

$\frac{dy}{y} = dt$

$\ln y = t + C_1$

$e^{\ln y} = e^{t + C_1}$

$\boxed{Y_h = Ce^{t}}$

Guess

$Y_p = \alpha e^{-2t}$

$\frac{dy}{dt} - y = e^{-2t}$

$\frac{d}{dt}(\alpha e^{-2t}) - \alpha e^{-2t} = e^{-2t}$

$-2\alpha e^{-2t} - \alpha e^{-2t} = e^{-2t}$

$-3\alpha e^{-2t} = e^{-2t}$

$\alpha = -\frac{1}{3}$

$\boxed{Y_p = -\frac{1}{3}e^{-2t}}$

$Y(t) = Y_h(t) + Y_p(t)$

$\boxed{Y(t) = Ce^{t} + -\frac{1}{3}e^{-2t}}$

(1ii) $\frac{dy}{dt} = -4y + 3e^{-t}$

associated homogeneous

$\frac{dy}{dt} = -4y$

$\frac{dy}{y} = -4\,dt$

$\ln y = -4t + C_1$

$e^{\ln y} = e^{-4t + C_1}$

$\boxed{Y_h = Ce^{-4t}}$

GUESS: $Y_p = \alpha e^{-t}$

$\frac{dy}{dt} + 4y = 3e^{-t}$

$\frac{d}{dt}(\alpha e^{-t}) + 4(\alpha e^{-t}) = 3e^{-t}$

$-\alpha e^{-t} + 4\alpha e^{-t} = 3e^{-t}$

$3\alpha e^{-t} = 3e^{-t}$

$\alpha = 1$

$\boxed{Y_p = e^{-t}}$

$Y(t) = Y_h(t) + Y_p(t)$

$\boxed{Y(t) = Ce^{-4t} + e^{-t}}$

(1 iii) $\frac{dy}{dt} = y + \cos 2t$

associated homogeneous

$$\frac{dy}{dt} = y$$

(we solved this in problem #1)

$$\boxed{y_h = Ce^y}$$

GUESS: $y_p = \alpha \cos 2t + \beta \sin 2t$

$$\frac{dy}{dt} - y = \cos 2t$$

$$\frac{d}{dt}(\alpha \cos 2t + \beta \sin 2t) - (\alpha \cos 2t + \beta \sin 2t) = \cos 2t$$

$$-2\alpha \sin 2t + 2\beta \cos 2t - \alpha \cos 2t + \beta \sin 2t = \cos 2t$$

$$(2\beta - \alpha) \cos 2t + (-2\alpha - \beta) \sin 2t = \cos 2t$$

want to be 1 want to be 0

$$2\beta - \alpha = 1$$

$$2(-\beta + -2\alpha = 0)$$

$\Rightarrow$

$$2\beta - \alpha = 1$$

$$-2\beta + -4\alpha = 0$$

$$-5\alpha = 1$$

$$\alpha = -\frac{1}{5}$$

$$2\beta - \left(-\frac{1}{5}\right) = 1$$

$$2\beta = 4/5$$

$$\boxed{\beta = 2/5}$$

$$\boxed{y_p = -\frac{1}{5} \cos 2t + \frac{2}{5} \sin 2t}$$

$$y(t) = y_h(t) + y_p(t)$$

$$\boxed{y(t) = Ce^y + (-)\frac{1}{5} \cos 2t + \frac{2}{5} \sin 2t}$$

(1 iv) $\frac{dy}{dt} = 2y + \sin 2t$

associated homogeneous

$$\frac{dy}{dt} = 2y$$

$$\frac{dy}{y} = 2\,dt$$

$$\ln y = 2t + C_1$$

$$e^{\ln y} = e^{2t + C_1}$$

$$\boxed{y_h = Ce^{2t}}$$

GUESS: $\alpha \cos 2t + \beta \sin 2t$

$$\frac{dy}{dt} - 2y = \sin 2t$$

$$\frac{d}{dt}\left[\alpha \cos 2t + \beta \sin 2t\right] - 2\left[\alpha \cos 2t + \beta \sin 2t\right] = \sin 2t$$

$$-2\alpha \sin 2t + 2\beta \cos 2t - (2\alpha \cos 2t + 2\beta \sin 2t) = \sin 2t$$

$$\underbrace{(2\beta - 2\alpha)}_{\text{want to be } 0} \cos 2t + \underbrace{(-2\alpha - 2\beta)}_{\text{want to be } 1} \sin 2t = \sin 2t$$

$$\begin{aligned} 2\beta - 2\alpha &= 0 \\ -2\alpha - 2\beta &= 1 \end{aligned} \implies \begin{aligned} -2\alpha + 2\beta &= 0 \\ -2\alpha - 2\beta &= 1 \\ \hline -4\alpha &= 1 \\ \alpha &= -\tfrac{1}{4} \end{aligned}$$

$$2\beta - 2\left(-\tfrac{1}{4}\right) = 0$$

$$2\beta = -\tfrac{2}{4}$$

$$\beta = -\tfrac{1}{4}$$

$$\boxed{y_p = -\tfrac{1}{4} \cos 2t - \tfrac{1}{4} \sin 2t}$$

$$y(t) = y_h(t) + y_p(t)$$

$$\boxed{y(t) = Ce^{2t} + \left(-\tfrac{1}{4}\right) \cos 2t + \left(-\tfrac{1}{4}\right) \sin 2t}$$

(1 v) $\frac{dy}{dt} = 4y - 5e^{4t}$

associated homogeneous

$$\frac{dy}{dt} = 4y$$

$$\frac{dy}{y} = 4\,dt$$

$$\ln y = 4t + C_1$$

$$e^{\ln y} = e^{4t + C_1}$$

$$\boxed{y_h = Ce^{4t}}$$

GUESS: $y_p = \alpha t e^{4t}$

we do this because αe^{4t} is merely the homogeneous solution.

$$\frac{dy}{dt} - 4y = -5e^{4t}$$

$$\frac{d}{dt}(\alpha t e^{4t}) - 4(\alpha t e^{4t}) = -5e^{4t}$$

$$\alpha e^{4t} + 4\alpha t e^{4t} - 4\alpha t e^{4t} = -5e^{4t}$$

$$\alpha e^{4t} = -5e^{4t}$$

$$\alpha = -5$$

$$\boxed{y_p = -5te^{4t}}$$

$$y(t) = y_h(t) + y_p(t)$$

$$\boxed{y(t) = Ce^{4t} + -5te^{4t}}$$

(1 vi) $\frac{dy}{dt} = \frac{y}{2} + 4e^{t/2}$

associated homogeneous

$$\frac{dy}{dt} = \frac{y}{2}$$

$$\frac{dy}{y} = \frac{1}{2}\,dt$$

$$\ln y = \frac{1}{2}t + C_1$$

$$e^{\ln y} = e^{\frac{1}{2}t + C_1}$$

$$\boxed{y_h = Ce^{t/2}}$$

GUESS: $y_p = \alpha t e^{t/2}$

we do this because $\alpha e^{t/2}$ is merely the homogeneous solution

$$\frac{dy}{dt} - \frac{y}{2} = 4e^{t/2}$$

$$\frac{d}{dt}\left(\alpha t e^{t/2}\right) - \frac{1}{2}\left(\alpha t e^{t/2}\right) = 4e^{t/2}$$

$$\alpha e^{t/2} + \frac{1}{2}\alpha t e^{t/2} - \frac{1}{2}\alpha t e^{t/2} = 4e^{t/2}$$

$$\alpha e^{t/2} = 4e^{t/2}$$

$$\alpha = 4$$

$$\boxed{y_p = 4te^{t/2}}$$

$$y(t) = y_h(t) + y_p(t)$$

$$\boxed{y(t) = Ce^{t/2} + 4te^{t/2}}$$

(2i) $\frac{dy}{dt} + 2y = e^{t/3}$; $y(0)=1$

associated homogeneous

$\frac{dy}{dt} = -2y$

$\frac{dy}{y} = -2\,dt$

$\ln y = -2t + c_1$

$e^{\ln y} = e^{-2t+c_1}$

$\boxed{y_h = Ce^{-2t}}$

GUESS: $y_p = \alpha e^{t/3}$

$\frac{dy}{dt} + 2y = e^{t/3}$

$\frac{d}{dt}(\alpha e^{t/3}) + 2(\alpha e^{t/3}) = e^{t/3}$

$\frac{1}{3}\alpha e^{t/3} + \frac{6}{3}\alpha e^{t/3} = e^{t/3}$

$\frac{7}{3}\alpha e^{t/3} = e^{t/3}$

$\frac{7}{3}\alpha = 1$

$\boxed{\alpha = 3/7}$

$\boxed{y_p = \frac{3}{7}e^{t/3}}$

$y(t) = Ce^{-2t} + \frac{3}{7}e^{t/3}$

$y(0) = 1$

$1 = Ce^{0} + \frac{3}{7}e^{0}$

$1 = c + \frac{3}{7}$

$\boxed{\frac{4}{7} = c}$

$\boxed{y(t) = \frac{4}{7}e^{-2t} + \frac{3}{7}e^{t/3}}$

(2ii) $\frac{dy}{dt} - 2y = 3e^{-2t}$; $y(0)=10$

associated homogeneos was solved in (1 iv)

$\boxed{y_h(t) = Ce^{2t}}$

GUESS: $y_p = \alpha e^{-2t}$

$\frac{dy}{dt} - 2y = 3e^{-2t}$

$\frac{d}{dt}(\alpha e^{-2t}) - 2(\alpha e^{-2t}) = 3e^{-2t}$

$-2\alpha e^{-2t} - 2\alpha e^{-2t} = 3e^{-2t}$

$-4\alpha e^{-2t} = 3e^{-2t}$

$-4\alpha = 3$

$\alpha = -\frac{3}{4}$

$\boxed{y_p = -\frac{3}{4}\alpha^{-2t}}$

$y(t) = Ce^{2t} - \frac{3}{4} \cdot e^{-2t}$

$y(0) = 10$

$10 = Ce^{0} - \frac{3}{4}e^{0}$

$10 = C - \frac{3}{4}$

$\boxed{\frac{43}{4} = c}$

$\boxed{y(t) = \frac{43}{4}e^{2t} - \frac{3}{4}e^{-2t}}$

(2 iii) $\frac{dy}{dt} + y = \cos 2t \quad ; \; y(0) = 5$

associated homogeneous

$$\frac{dy}{dt} = -y$$

$$\frac{dy}{y} = -dt$$

$$\ln y = -t + C_1$$

$$e^{\ln y} = e^{-t + C_1}$$

$$\boxed{y = Ce^{-t}}$$

GUESS: $y_p = \alpha \cos 2t + \beta \overset{\sin}{\cancel{\cos}}\, 2t$

$$\frac{dy}{dt} + y = \cos 2t$$

$$\frac{d}{dt}(\alpha \cos 2t + \beta \overset{\sin}{\cancel{\cos}}\, 2t) + (\alpha \cos 2t + \beta \overset{\sin}{\cancel{\cos}}\, 2t) = \cos 2t$$

$$-2\alpha \sin 2t + 2\beta \cos 2t + \alpha \cos 2t + \beta \sin 2t = \cos 2t$$

$$(\alpha + 2\beta) \cos 2t + (-2\alpha + \beta) \sin 2t = \cos 2t$$

$\uparrow$ want to be 1 $\qquad\qquad$ $\uparrow$ want to be 0

$$\begin{aligned} \alpha + 2\beta &= 1 \\ -2[-2\alpha + \beta &= 0] \end{aligned} \implies \begin{aligned} \alpha + 2\beta &= 1 \\ 4\alpha - 2\beta &= 0 \\ \hline 5\alpha &= 1 \\ \alpha &= \tfrac{1}{5} \end{aligned}$$

$$\frac{1}{5} + 2\beta = 1$$

$$2\beta = 4/5$$

$$\beta = 2/5 \qquad\qquad y_p = \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t$$

$$y(t) = Ce^{-t} + \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t$$

$y(0) = 5$

$$5 = Ce^{0} + \frac{1}{5}\cos(0) + \frac{2}{5}\sin(0)$$

$$5 = C + \frac{1}{5}\cdot(1) + \frac{2}{5}\cdot(0)$$

$$5 = C + \frac{1}{5}$$

$$\boxed{\frac{24}{5} = C} \qquad \boxed{y(t) = \frac{24}{5}e^{-t} + \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t}$$

(2 iv) $\frac{dy}{dt} + y = 3\cos 2t$; $y(0) = -1$

associated homogeneous was solved in last problem

$$\boxed{y_h = Ce^{-t}}$$

GUESS: $y_p = \alpha \cos 2t + \beta \sin 2t$

$$\frac{dy}{dt} + y = 3\cos 2t$$

$$\frac{d}{dt}(\alpha \cos 2t + \beta \sin 2t) + (\alpha \cos 2t + \beta \sin 2t) = 3\cos 2t$$

$$-2\alpha \sin 2t + 2\beta \cos 2t + \alpha \cos 2t + \beta \sin 2t = 3\cos 2t$$

$$\underbrace{(3\alpha + 2\beta)}_{\text{want to be } 1} \cos 2t + \underbrace{(-2\alpha + 3\beta)}_{\text{want to be } 0} \sin 2t = 3\cos 2t$$

$$\begin{aligned} 2[3\alpha + 2\beta = 1] \\ 3[-2\alpha + 3\beta = 0] \end{aligned} \Longrightarrow \begin{aligned} 6\alpha + 4\beta &= 2 \\ -6\alpha + 9\beta &= 0 \\ \hline 13\beta &= 2 \end{aligned}$$

$$\boxed{\beta = 2/13}$$

$$3\alpha + 2\left(\tfrac{2}{13}\right) = 1$$

$$3\alpha + \tfrac{4}{13} = 1$$

$$3\alpha = 9/13$$

$$\boxed{\alpha = 3/13}$$

$$\boxed{y_p = \frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t}$$

$$y(t) = Ce^{-t} + \frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t$$

$y(0) = -1$

$$-1 = Ce^{0} + \tfrac{3}{13}\cos(0) + \tfrac{2}{13}\sin(0)$$

$$-1 = C + \tfrac{3}{13} + 0$$

$$-1 = C + 3/13$$

$$\boxed{C = -16/13}$$

$$\boxed{y(t) = -\frac{16}{13}e^{-t} + \frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t}$$

(2 v) $\frac{dy}{dt} + 5y = 3e^{-5t}$; $y(0) = -2$

associated homogeneous

$$\frac{dy}{dt} = -5y$$

$$\frac{dy}{y} = -5dt$$

$$\ln y = -5t + C_1$$

$$e^{\ln y} = e^{-5t + C_1}$$

$$\boxed{y_h = Ce^{-5t}}$$

GUESS: $\alpha t e^{-5t}$ because αe^{-5t} is the homogeneous solution

$$\frac{d}{dt} + 5y = 3e^{-5t}$$

$$\frac{d}{dt}(\alpha t e^{-5t}) + 5(\alpha t e^{-5t}) = 3e^{-5t}$$

$$\alpha e^{-5t} + (-)5\alpha t e^{-5t} + 5\alpha t e^{-5t} = 3e^{-5t}$$

$$\alpha e^{-5t} = 3e^{-5t}$$

$$\boxed{\alpha = 3}$$

$$\boxed{y_p = 3te^{-5t}}$$

$$y(t) = Ce^{-5t} + 3te^{-5t}$$

$$y(0) = -2$$

$$-2 = Ce^{0} + 3(0)$$

$$\boxed{-2 = c}$$

$$\boxed{\begin{gathered} y(t) = -2e^{-5t} + 3te^{-5t} \\ \text{OR} \\ y(t) = (-2 + 3t)e^{-5t} \end{gathered}}$$

(2 vi) $\frac{dy}{dt} - 2y = 7e^{2t}$; $y(0) = 3$

8

associated homogeneous was solved previously

$$\boxed{y_h = Ce^{2t}}$$

GUESS: $\alpha t e^{2t}$ because αe^{-5t} is the homogeneous solution

$$\frac{dy}{dt} - 2y = 7e^{2t}$$

$$\frac{d}{dt}(\alpha t e^{2t}) - 2(\alpha t e^{2t}) = 7e^{2t}$$

$$2\alpha e^{2t} + 2\alpha t e^{2t} - 2\alpha t e^{2t} = 7e^{2t}$$

$$2\alpha e^{2t} = 7e^{2t}$$

$$2\alpha = 7$$

$$\boxed{\alpha = \frac{7}{2}}$$

$$\boxed{y_p = \frac{7}{2}te^{2t}}$$

$$y(t) = Ce^{2t} + \frac{7}{2}te^{2t}$$

$$y(0) = 3$$

$$3 = Ce^{0} + \frac{7}{2}(0)e^{0}$$

$$\boxed{3 = c}$$

$$\boxed{\begin{gathered} y(t) = 3e^{2t} + \frac{7}{2}te^{2t} \\ \text{OR} \\ y(t) = \left(3 + \frac{7}{2}t\right)e^{2t} \end{gathered}}$$